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Applied Stochastic Processes

Solution sheet 1

Solution 1.1 [Markov chains]

i) (a) The stochastic process $(X_n)_{n\geq 1}$ takes values in $S = \mathbb{N} \cup \{\infty\}$. Here, we denote by ∞ the state representing that no six has been rolled so far.

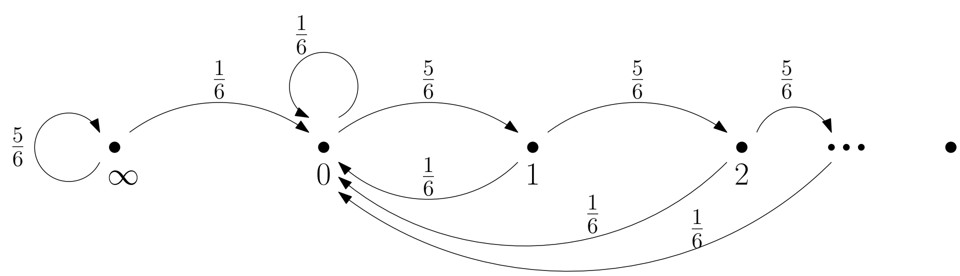
(b) X is a Markov chain, as shown in (c).

(c) We determine the initial distribution μ and a transition probability P such that $X \sim \mathrm{MC}(\mu, P)$. The transition probability $P = (p_{ij})_{i,j\in S}$ is given by

$$p_{ij} = \begin{cases} \frac{1}{6} & \text{if } j = 0, \\ \frac{5}{6} & \text{if } j = i+1 \text{ or } j = i = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, we have $X_n = 0$ if and only if $\xi_n = 6$, which happens with probability 1/6. If $\xi_n \neq 6$, which happens with probability 5/6, we have $X_n = X_{n-1} + 1$ (respectively $X_n = X_n - 1 = \infty$). The initial distribution of X_1 is $\mu = 1/6 \cdot \delta^0 + 5/6 \cdot \delta^\infty$. Alternatively, we can also see the process as starting at $X_0 = \infty$, i.e. having $\mu = \delta^\infty$.

(d) We can represent P by the following weighted graph.



(e) For every $n \geq 1$, we have

$$p_{\infty\infty}^{(n)} = \left(\frac{5}{6}\right)^n \quad \text{and} \quad p_{i\infty}^{(n)} = 0, \ \forall i \in \mathbb{N}.$$

For every $j \in \mathbb{N}$, $i \in \mathbb{N} \cup \{\infty\}$, and every $n \geq 1$, we have

$$p_{ij}^{(n)} = \left(\frac{5}{6}\right)^n \quad \text{if } n = j - i,$$
$$p_{ij}^{(n)} = \frac{1}{6} \cdot \left(\frac{5}{6}\right)^j \quad \text{if } j \leq n-1.$$

ii) (a) The stochastic process $(X_n)_{n\geq 1}$ takes values in $S = \{1, \dots, 6\}$.

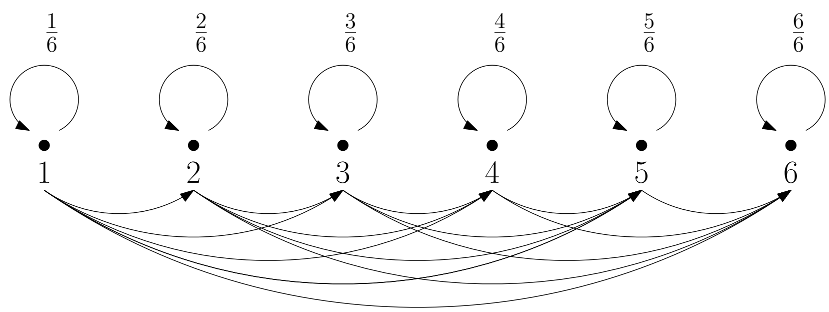
(b) X is a Markov chain, as shown in (c).

(c) We determine the initial distribution μ and a transition probability P such that $X \sim \mathrm{MC}(\mu, P)$. The transition probability $P = (p_{ij})_{i,j\in S}$ is given by

$$p_{ij} = \begin{cases} 0 & \text{if } j < i, \\ \frac{i}{6} & \text{if } j = i, \\ \frac{1}{6} & \text{if } j > i. \end{cases}$$

Indeed, if $X_{n-1} = i$, then we have $X_n = i$ if and only if $\xi_n \leq i$, which happens with probability $i/6$, and we have $X_n = j$ for $j > i$ if and only if $\xi_n = j$, which happens with probability $1/6$. The initial distribution of X_1 is uniform on S, i.e. $\mu = 1/6 \cdot \left(\delta^1 + \ldots + \delta^6\right)$.

(d) We can represent P by the following weighted graph.



Here, all weights on the directed edges (i, j) with $j > i$ are equal to $1/6$.

(e) For every $i, j \in \{1, \ldots, 6\}$ and every $n \geq 1$, we have

$$\begin{aligned} p_{ij}^{(n)} &= 0 \quad \text{if } j < i, \\ p_{ij}^{(n)} &= \left(\frac{i}{6}\right)^n \quad \text{if } j = i, \\ p_{ij}^{(n)} &= \left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n \quad \text{if } j > i. \end{aligned}$$

iii) (a) The stochastic process $(X_n)_{n\geq 1}$ takes values in $S = \{1, \ldots, 6\}$.

(b) X is a not Markov chain. We note that

$$\{X_2 = 1, X_3 = 6\} = \{X_2 = 1, X_3 = 6, X_4 = 6\} = \{\xi_1 = 1, \xi_2 = 1, \xi_3 = 6\},$$

and so

$$\mathbb{P}[X_2 = 1, X_3 = 6] = \mathbb{P}[X_2 = 1, X_3 = 6, X_4 = 6] = (1/6)^3.$$

If $X \sim \mathrm{MC}(\mu, P)$ for some initial distribution μ and transition probability P, then it would follow from Definition 1.3 that

$$p_{66} = \frac{\mathbb{P}[X_2 = 1, X_3 = 6, X_4 = 6]}{\mathbb{P}[X_2 = 1, X_3 = 6]} = 1, \quad \text{thus } p_{66}^{(n)} = 1, \ \forall n \geq 1.$$

But this contradicts the definition of X since the stochastic process can leave the state 6 with positive probability.

iv) (a) The stochastic process $(X_n)_{n\geq 1}$ takes values in $S = \{1, \ldots, 6\}$.

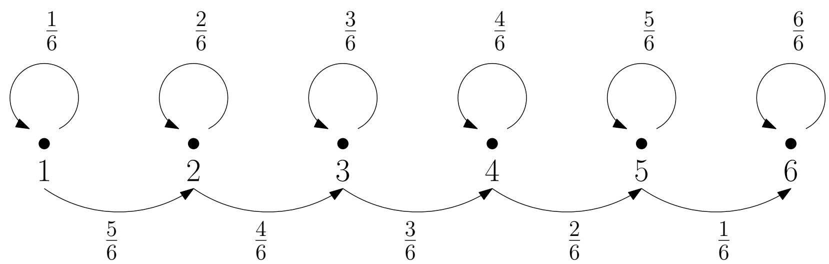
(b) X is a Markov chain, as shown in (c).

(c) We determine the initial distribution μ and a transition probability P such that $X \sim \mathrm{MC}(\mu, P)$. The transition probability $P = (p_{ij})_{i,j\in S}$ is given by

$$p_{ij} = \begin{cases} \frac{6-i}{6} & \text{if } j = i+1, \\ \frac{i}{6} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, if $X_{n-1} = i$, then we have $X_n = i+1$ if and only if ξ_n takes a new value, which happens with probability $(6-i)/6$, and we have $X_n = i$ if and only if ξ_n takes no new value, which happens with probability $i/6$. The initial distribution of X_1 is $\mu = \delta^1$.

(d) We can represent P by the following weighted graph.



(e) In this case, it requires a bit more work to determine the n-step transition probabilities. We proceed by diagonalizing the matrix P of the transition probability, given by

$$P = \begin{pmatrix} 1/6 & 5/6 & 0 & 0 & 0 & 0 \\ 0 & 2/6 & 4/6 & 0 & 0 & 0 \\ 0 & 0 & 3/6 & 3/6 & 0 & 0 \\ 0 & 0 & 0 & 4/6 & 2/6 & 0 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 6/6 \end{pmatrix}.$$

Since it is an upper triangular matrix, its eigenvalues are equal to the diagonal entries. By computing the associated eigenvectors, we obtain the matrix Q with the right eigenvectors as columns, given by

$$Q = \begin{pmatrix} 1 & 5 & 10 & 10 & 5 & 1 \\ 0 & 1 & 4 & 6 & 4 & 1 \\ 0 & 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad Q^{-1} = \begin{pmatrix} 1 & -5 & 10 & -10 & 5 & -1 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -3 & 3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$P^n = Q \cdot \begin{pmatrix} (1/6)^n & 0 & 0 & 0 & 0 & 0 \\ 0 & (2/6)^n & 0 & 0 & 0 & 0 \\ 0 & 0 & (3/6)^n & 0 & 0 & 0 \\ 0 & 0 & 0 & (4/6)^n & 0 & 0 \\ 0 & 0 & 0 & 0 & (5/6)^n & 0 \\ 0 & 0 & 0 & 0 & 0 & (6/6)^n \end{pmatrix} \cdot Q^{-1},$$

which allows to deduce all transition probabilities.

Solution 1.2 [Deterministic Markov chains]

(a) A deterministic sequence $(x_n)_{n\geq 0}$ is a Markov chain if and only if there exists a function $\Phi : S \to S$ such that for all $n \geq 0$, $x_{n+1} = \Phi(x_n)$.

$(\Longleftarrow)$: It follows directly that $(x_n)_{n\geq 0}$ is a Markov chain $\mathrm{MC}(\mu, P)$ with $\mu = \delta^{x_0}$ and transition probability P given by $p_{ij} = \mathbb{1}_{j=\Phi(i)}$.

$(\Longrightarrow)$: Given $(x_n)_{n\geq 0}$, we define $\Phi : S \to S$ by

$$\Phi(x) = \begin{cases} x_{n+1} & \text{if } \exists n \geq 0 \text{ s.t. } x_n = x, \\ x & \text{if } \forall n \geq 0,\ x_n \neq x. \end{cases}$$

Let $x, y \in S$. The function Φ is well-defined since for every $n \geq 0$ with $x_n = x$,

$$p_{xy} = \frac{\mathbb{P}[x_{n+1} = y, x_n = x]}{\mathbb{P}[x_n = x]} = \mathbb{1}_{x_{n+1}=y},$$

where we used Definition 1.3 in the first inequality and the fact that the sequence is deterministic in the second inequality.

(b) There are three possible choices for the initial distribution, δ^1, δ^2, δ^3. Using the previous exercise, we can choose $\Phi(i) \in \{1, 2, 3\}$ for every $i \in \{1, 2, 3\}$, i.e. there are $3^3 = 27$ possible choices for the transition probability P.

Solution 1.3 [Matrix representation]

(a) Our goal is to show that for any $k \geq 1$, $n_k \geq \ldots \geq n_1 \geq 0$, and $i_1, \ldots, i_k \in S$,

$$\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k] = \left(\delta^i P^{n_1} \delta_{\{i_1\}} P^{n_2-n_1} \delta_{\{i_2\}} \cdots P^{n_k-n_{k-1}}\right)_{i_k}.$$

Since

$$\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k] = \sum_{i_0=1}^{N} \delta^i(i_0) p_{i_0 i_1}^{(n_1)} p_{i_1 i_2}^{(n_2-n_1)} \cdots p_{i_{k-1} i_k}^{(n_k-n_{k-1})},$$

it is sufficient to show that for any $k \geq 1$, $n_k \geq \ldots \geq n_1 \geq 0$, and $i_0, i_1, \ldots, i_k \in S$,

$$\left(P^{n_1} \delta_{\{i_1\}} P^{n_2-n_1} \delta_{\{i_2\}} \cdots P^{n_k-n_{k-1}}\right)_{i_0, i_k} = p_{i_0 i_1}^{(n_1)} p_{i_1 i_2}^{(n_2-n_1)} \cdots p_{i_{k-1} i_k}^{(n_k-n_{k-1})}$$

We proceed by induction on k. For $k = 1$,

$$\left(P^{n_1}\right)_{i_0, i_1} = p_{i_0 i_1}^{(n_1)},$$

by Definition 1.4. For $k \geq 2$, we use the induction hypothesis on $k-1$ to obtain

$$\begin{aligned}
&\left(P^{n_1} \delta_{\{i_1\}} P^{n_2-n_1} \delta_{\{i_2\}} \cdots P^{n_k-n_{k-1}}\right)_{i_0, i_k} \\
&= \sum_{j=1}^{N} \left(P^{n_1} \delta_{\{i_1\}} P^{n_2-n_1} \delta_{\{i_2\}} \cdots P^{n_{k-1}-n_{k-2}}\right)_{i_0, j} \cdot \left(\delta_{i_{k-1}} P^{n_k-n_{k-1}}\right)_{j, i_k} \\
&= \sum_{j=1}^{N} p_{i_0 i_1}^{(n_1)} p_{i_1 i_2}^{(n_2-n_1)} \cdots p_{i_{k-2} j}^{(n_{k-1}-n_{k-2})} \cdot \left(P^{n_k-n_{k-1}}\right)_{j, i_k} \mathbb{1}_{j=i_{k-1}} \\
&= p_{i_0 i_1}^{(n_1)} p_{i_1 i_2}^{(n_2-n_1)} \cdots p_{i_{k-1} i_k}^{(n_k-n_{k-1})}
\end{aligned}$$

as desired.

(b) Our goal is to show that for any $k \geq 1$, $n_k \geq \ldots \geq n_1 \geq 0$, and $A_1, \ldots, A_k \subseteq S$,

$$\mathbb{P}[X_{n_1} \in A_1, X_{n_2} \in A_2, \ldots, X_{n_k} \in A_k] = \sum_{i_k \in A_k} \left(\delta^i \S P^{n_1} \delta_{A_1} P^{n_2-n_1} \delta_{A_2} \cdots P^{n_k-n_{k-1}}\right)_{i_k}.$$

This can be established analogously to the previous exercise.

Solution 1.4 [**n-step transition probability**]
Let us identify the set a, b, c with $1, 2, 3$. Then, from the diagram we can get the following transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

We know that $p^{(n)}_{a,a} = P^n(1,1)$. Then we need to calculate P^n. We see that this matrix is diagonalizable since it has different eigenvalues. Indeed, it characteristic equation is given by

$$0 = \det(\lambda I - P) = \lambda \left(\lambda - \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{1}{4}(\lambda - 1)(4\lambda^2 + 1)$$

and its eigenvalues are $1, i/2, -i/2$. Hence, there exists an invertible matrix U such that

$$P = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix} U^{-1}$$

and then

$$P^n = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & (i/2)^n & 0 \\ 0 & 0 & (-i/2)^n \end{pmatrix} U^{-1}$$

This implies that $P^n(1,1) = x + y(i/2)^n + z(-i/2)^n$ for some constants x, y, z. We can calculate the value of these constants by using the first steps of our chain

$$\begin{aligned} 1 &= P^0(1,1) = x + y + z \\ 0 &= P^1(1,1) = x + iy/2 - iz/2 \\ 0 &= P^2(1,1) = x - y/4 - z/4. \end{aligned}$$

This give us $x = 1/5$, $y = (i-2)/5$ and $z = (2-i)/5$. Therefore

$$\begin{aligned} P^n(1,1) &= \frac{1}{5} + \frac{i-2}{5}\left(\frac{i}{2}\right)^n + \frac{2-i}{5}\left(\frac{-i}{2}\right)^n \\ &= \frac{1}{5} + \frac{i-2}{5}\left(\frac{1}{2}\right)^n \left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right) + \frac{2-i}{5}\left(\frac{1}{2}\right)^n \left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right) \\ &= \frac{1}{5} + \left(\frac{1}{2}\right)^n \left(\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right). \end{aligned}$$

Solution 1.5 [Construction of random variables]

(a) By definition of the infimum and using right continuity of the distribution function F, we have for every $x \in \mathbb{R}$ and $\alpha \in (0,1)$,

$$(F^{-1}(\alpha) \leq x) \iff (\alpha \leq F(x)).$$

The generalized inverse $F^{-1} : (0,1) \to \mathbb{R}$ of a distribution function is measurable, and so we can define the random variable $X := F^{-1}(U)$. Clearly,

$$\mathbb{P}[X \leq x] = \mathbb{P}[F^{-1}(U) \leq x] = \mathbb{P}[U \leq F(x)] = F(x).$$

(b) Let $U \sim \mathcal{U}([0,1])$. We consider the binary expansion $0.Y_1 Y_2 \dots$ of U, which can be defined as

$$Y_1 := \lfloor 2U \rfloor,$$

and inductively for $i \geq 2$,

$$Y_i := \lfloor 2^i U - \sum_{j=1}^{i-1} 2^j Y_j \rfloor.$$

Thus,

$$U = \sum_{i=1}^{\infty} Y_i 2^{-i}.$$

For all $N \geq 1$ and for all $y_1, \dots, y_N \in \{0,1\}$, we have

$$\mathbb{P}[Y_1 = y_1, \dots, Y_N = y_N] = 2^{-N} = P[Y_1 = y_1] \cdot \dots \cdot \mathbb{P}[Y_N = y_N],$$

since $\mathbb{P}[U \in [i \cdot 2^{-N}, (i+1) \cdot 2^{-N})] = 2^{-N}$ for every $i \in \{0, \dots, 2^N - 1\}$. Thus, for all $N \geq 1$, $(Y_i)_{i=1}^N$ is a sequence of independent, Bernoulli(1/2)-distributed random variables. A standard application of Dynkin's lemma implies that $(Y_i)_{i \geq 1}$ is a sequence of independent, Bernoulli(1/2)-distributed random variables.

(c) In (b), we have constructed a sequence $(Y_i)_{i \geq 1}$ of independent Bernoulli(1/2)-distributed random variables. It now suffices to choose an injective map $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (e.g. $\phi(i,j) := 2^i \cdot 3^j$) and to define

$$X_{i,j} := Y_{\phi(i,j)}.$$

In this way, $(X_{i,j})_{i,j \geq 1}$ is a family of independent, Bernoulli(1/2)-distributed random variables. We then define for every $i \geq 1$,

$$U_i := \sum_{j=1}^{\infty} 2^{-j} X_{i,j}.$$

(d) Analogously to (a), we define for every $i \geq 1$,

$$X_i = F_i^{-1}(U_i).$$