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Applied Stochastic Processes

Solution sheet 10

Solution 10.1

(a) On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we define the measure μ by

$$\mu(B) = \begin{cases} +\infty & \text{if } 0 \in B, \\ 0 & \text{otherwise,} \end{cases}$$

for $B \in \mathcal{B}(\mathbb{R})$. The measure μ is *not* σ-finite since for any family $(B_i)_{i\geq 1} \subset \mathcal{B}(\mathbb{R})$ with $\mu(B_i) < \infty$ for all $i \geq 1$, it holds that $0 \notin \bigcup_{i\geq 1} B_i$.

(b) By assumption, U is a random variable taking values in $[0, 5]$. Therefore, the random variable

$$\delta_U : \begin{cases} \Omega & \to \mathcal{M} \\ \omega & \mapsto \delta_{U(\omega)} \end{cases}$$

is well-defined since for all $u \in [0, 5]$, δ_u is a σ-finite measure on $([0, 5], \mathcal{B}([0, 5]))$ taking values in $\{0, 1\}$. Hence, δ_U is point process on $([0, 5], \mathcal{B}([0, 5]))$.

(c) For all $u \in [0, 5]$, $2 \cdot \delta_u$ is a σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ taking values in $\{0, 2\}$. As in (b), we deduce that $2 \cdot \delta_U$ is a point process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(d) δ_U is not a Poisson point process on $([0, 5], \mathcal{B}([0, 5]))$ since $\delta_U([0, 1])$ and $\delta_U([4, 5])$ are not independent.

(e) By the superposition theorem from Section 6.6, the process $M_1 + M_2$ is a Poisson point process if the measure $\mu := \mu_1 + \mu_2$ is σ-finite. It therefore suffices to note that the σ-finiteness of μ follows from the σ-finiteness of μ_1 and μ_2. Indeed, let $(A_i)_{i\geq 1} \subset \mathcal{E}$ and $(B_i)_{i\geq 1} \subset \mathcal{E}$ be increasing sequences such that $\bigcup_{i\geq 1} A_i = \bigcup_{i\geq 1} B_i = E$, and $\mu_1(A_i) < \infty$ and $\mu_2(B_i) < \infty$ for all $i \geq 1$. Then $(A_i \cap B_i)_{i\geq 1}$ is an increasing sequence satisfying for all $i \geq 1$,

$$\mu(A_i \cap B_i) = \underbrace{\mu_1(A_i \cap B_i)}_{\leq \mu_1(A_i)} + \underbrace{\mu_2(A_i \cap B_i)}_{\leq \mu_2(B_i)} < \infty,$$

and $\bigcup_{i\geq 1}(A_i \cap B_i) = E$.

(f) By the superposition theorem from Section 6.6, the process $\sum_{i=1}^{\infty} M_i$ is a Poisson point process if the measure $\mu := \sum_{i=1}^{\infty} \mu_i$ is σ-finite. Even though the measures μ_i, $i \geq 1$, are all σ-finite, this is not necessarily the case. For example, if $\mu_i := \delta_0$ for all $i \geq 1$, then the measure μ is not σ-finite as shown in (a).

Solution 10.2

(a) First, we note that the measure $\mu \otimes \nu$ is σ-finite. Indeed, choose a partition $(E_i)_{i\geq 1}$ of $\mathbb{R}$ such that E_i measurable and $\mu(E_i) < \infty$ for every i, and then consider the partition $(E_i \times \{j\})_{i\geq 1, j\in\{0,1\}}$, which satisfies

$$\mu \otimes \nu(E_i \times \{j\}) = \mu(E_i) \cdot \underbrace{\nu(\{j\})}_{=1} = \mu(E_i) < \infty.$$

It now follows from the superposition theorem in Section 6.6 that $M_0 + M_1$ is a Poisson point process on $\mathbb{R} \times \{0,1\}$ with intensity $\mu_0 + \mu_1$ and it suffices to note that $\mu_0 + \mu_1 = \mu \otimes \nu$.

(b) We will show that the process $\widetilde{M}$ is *not* a Poisson point process on $\mathbb{R} \times \{0,1\}$.

As in part (a), let $(E_i)_{i\geq 1}$ be a partition of $\mathbb{R}$ such that E_i measurable and $\mu(E_i) < \infty$ for every i. Without loss of generality, assume that $\mu(E_i) > 0$ for every i. We compute

$$\begin{aligned}
&\mathbb{P}[\widetilde{M}(E_1 \times \{0\}) = 1, \widetilde{M}(E_1 \times \{1\}) = 1, \widetilde{M}(E_2 \times \{0\}) = 1, \widetilde{M}(E_2 \times \{1\}) = 0] \\
&= \mathbb{P}[M(E_1) \cdot M'(\{0\}) = 1, M(E_1) \cdot M'(\{1\}) = 1, M(E_2) \cdot M'(\{0\}) = 1, M(E_2) \cdot M'(\{1\}) = 0] \\
&= 0,
\end{aligned}$$

since the first three events require $M(E_1) = M(E_2) = 1$ and $M'(\{0\}) = M'(\{1\}) = 1$ but the fourth event requires $M(E_2) = 0$ or $M'(\{1\}) = 0$.

However, note that the sets $E_1 \times \{0\}$, $E_1 \times \{1\}$, $E_2 \times \{0\}$ and $E_2 \times \{1\}$ are disjoint, and so if $\widetilde{M}$ would be a Poisson point process, independence would imply

$$\begin{aligned}
&\mathbb{P}[\widetilde{M}(E_1 \times \{0\}) = 1, \widetilde{M}(E_1 \times \{1\}) = 1, \widetilde{M}(E_2 \times \{0\}) = 1, \widetilde{M}(E_2 \times \{1\}) = 0] \\
&= \mathbb{P}[\widetilde{M}(E_1 \times \{0\}) = 1] \cdot \mathbb{P}[\widetilde{M}(E_1 \times \{1\}) = 1] \cdot \mathbb{P}[\widetilde{M}(E_2 \times \{0\}) = 1] \cdot \mathbb{P}[\widetilde{M}(E_2 \times \{1\}) = 0] \\
&= \mathbb{P}[M(E_1) = M'(\{0\}) = 1] \cdot \mathbb{P}[M(E_1) = M'(\{1\}) = 1] \\
&\quad \cdot \mathbb{P}[M(E_2) = M'(\{0\}) = 1] \cdot \mathbb{P}[M(E_2) \cdot M'(\{1\}) = 0] \\
&> 0
\end{aligned}$$

since $\mu(E_1), \mu(E_2) \in (0,\infty)$ and $\nu(\{0\}) = \nu(\{1\}) = 1 \in (0,\infty)$. Hence, $\widetilde{M}$ cannot be a Poisson point process.

Solution 10.3

(a) Let $n \geq 0$. Using the independence of $X_1, \ldots, X_k$ in the first equality and their Poisson distribution in the second equality, we obtain

$$\begin{aligned}
\mathbb{P}[X_1 + \ldots + X_k = n] &= \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ \text{s.t. } i_1+\ldots+i_k = n}} \mathbb{P}[X_1 = i_1] \cdots \mathbb{P}[X_k = i_k] \\
&= e^{-(\lambda_1+\ldots+\lambda_k)} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ \text{s.t. } i_1+\ldots+i_k = n}} \frac{{\lambda_1}^{i_1}}{i_1!} \cdots \frac{{\lambda_k}^{i_k}}{i_k!} \\
&= e^{-(\lambda_1+\ldots+\lambda_k)} \frac{1}{n!} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ \text{s.t. } i_1+\ldots+i_k = n}} \binom{n}{i_1,\ldots,i_k} {\lambda_1}^{i_1} \cdots {\lambda_k}^{i_k} \\
&= e^{-(\lambda_1+\ldots+\lambda_k)} \frac{(\lambda_1 + \ldots + \lambda_k)^n}{n!},
\end{aligned}$$

which shows that $X_1 + \ldots + X_k \sim \text{Pois}(\lambda_1 + \ldots \lambda_k)$.

(b) For $k \geq 1$, define the partial sums $\bar{X}_k := \sum_{i=1}^k X_i$. We first note that $(\bar{X}_k)_{k\geq 1}$ is almost surely a monotone sequence and thus converges almost surely. Hence, $\bar{X}_\infty := \sum_{i=1}^\infty X_i$ is a well-defined random variable taking values in $\mathbb{N} \cup \{+\infty\}$, and we are left with determining its distribution.

Case 1: $\lambda = \sum_{i=1}^\infty \lambda_i = \infty$. In this case, a union bound implies that

$$\mathbb{P}[\bar{X}_\infty < \infty] = \mathbb{P}[\exists I \geq 1, \forall i > I : X_i = 0] \leq \sum_{I\geq 1} \mathbb{P}[\forall i > I : X_i = 0] = \sum_{I \geq 1} \exp(-\underbrace{\sum_{i>I} \lambda_i}_{=\infty}) = 0.$$

Hence, $\bar{X}_\infty = \infty$ almost surely.

Case 2: $\lambda = \sum_{i=1}^\infty \lambda_i < \infty$. From part (a), we know that $\bar{X}_k$ is Poisson-distributed with parameter $\sum_{i=1}^k \lambda_i$. Hence, for all $n \geq 0$,

$$\mathbb{P}[\bar{X}_k = n] = \exp\big(-\sum_{i=1}^k \lambda_i\big) \cdot \frac{(\sum_{i=1}^n \lambda_i)^n}{n!} \longrightarrow \exp(-\lambda) \cdot \frac{\lambda^n}{n!} \quad \text{as } k \to \infty,$$

and so, $\bar{X}_\infty$ is Pois(λ)-distributed.

Solution 10.4 Let us first consider $u(x) = \mathbf{1}_B(x)$ for some $B \in \mathcal{E} = \mathcal{B}(\mathbb{R})$.

(a) As $u(x) = \mathbf{1}_B(x)$, we have $\int u(x)M(dx) = M(B)$ which is a well-defined random variable according to the definition of a point process (Definition 6.1)

(b) Moreover, we then have $\mathbb{E}[\int u(x)M(dx)] = \mathbb{E}[M(B)] = \mu(B) = \int u(x)\mu(dx)$.

By linearity we can extend both results to simple functions. Since the limit of measurable functions is measurable and using monotone convergence theorem, we can also extend both results to arbitrary $u : E \to \mathbb{R}_0^+$. Now let us consider $u : E \to \mathbb{R}$. We can write $u = u_+ - u_-$ with $u_+, u_- : E \to \mathbb{R}_0^+$, and this implies that $\int u(x)M(dx)$ is a well-defined random variable. Assume that $\int |u(x)|\mu(dx) < \infty$. Then we also have that

$$\begin{aligned}\mathbb{E}\left[\int u(x)M(dx)\right] &= \mathbb{E}\left[\int u_+(x)M(dx)\right] - \mathbb{E}\left[\int u_-(x)M(dx)\right] \\ &= \int u_+(x)\mu(dx) - \int u_-(x)\mu(dx) = \int u(x)\mu(dx),\end{aligned}$$

which concludes the proof.