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Applied Stochastic Processes

Solution sheet 11

Solution 11.1

(a) For $B \in \mathcal{B}(\mathbb{R})$,
$$T\#\mu(B) = \mu\left(T^{-1}(B)\right) = \mu(B \times [0,2]) = 2 \cdot \mathrm{Leb}_{\mathbb{R}}(B).$$
Hence, $T\#\mu$ is σ-finite and $T\#M$ is a Poisson point process on $\mathbb{R}$ with intensity measure $T\#\mu = 2 \cdot \mathrm{Leb}$.

(b) For $B \in \mathcal{B}(\mathbb{R})$,
$$T\#\mu(B) = \mu(B \times \mathbb{R}) = \begin{cases} 0 & \text{if } \mathrm{Leb}_{\mathbb{R}}(B) = 0, \\ \infty & \text{if } \mathrm{Leb}_{\mathbb{R}}(B) > 0. \end{cases}$$
Hence, $T\#\mu$ is not σ-finite and $T\#M$ is a not a Poisson point process on $\mathbb{R}$.

(c) For $B \in \mathcal{B}([0,1])$,
$$T\#\mu(B) = 4 \cdot \mathrm{Leb}_{[0,1]}(B).$$
Hence, $T\#\mu$ is σ-finite and $T\#M$ is a Poisson point process on $[0,1]$ with intensity measure $T\#\mu = 4 \cdot \mathrm{Leb}$.

(d) We note that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathcal{C}^1$-diffeomorphism and $T^{-1}(y_1, y_2) = (y_2/2, y_1/2)$. Therefore, for all $(y_1, y_2) \in \mathbb{R}^2$, $|\det(dT^{-1}(y_1, y_2))| = 1/4$. Hence, $T\#\mu = 1/4 \cdot \mathrm{Leb}$ is σ-finite and $T\#M$ is a Poisson point process on $\mathbb{R}$ with intensity measure $T\#\mu = 1/4 \cdot \mathrm{Leb}$.

(e) By the restriction theorem, the restricted processes $M_{[0,1]^2}$, $M_{[0,2]^2}$, and $M_{[2,3]^2}$ are Poisson point processes with intensity measure Leb (on the subsets). Again by the restriction theorem, $M_{[0,1]^2}$ is independent of $M_{[2,3]^2}$ and $M_{[0,2]^2}$ is independent of $M_{[2,3]^2}$ (note that $M(\{(2,2)\}) = 0$ a.s.). The restricted processes $M_{[0,1]^2}$ and $M_{[0,2]^2}$ are *not* independent. For example, it can be seen by noticing that $M_{[0,2]^2}([0,2]^2) = 0$ implies $M_{[0,1]^2}([0,1]^2) = 0$.

Solution 11.2

(a) We consider the map $T : \mathbb{R}^d \to [0, \infty)$ defined by $T(x) = \|x\|_2 = \sqrt{x_1^2 + \cdots + x_d^2}$, which is a continuous function, and so it is measurable. Since $T\#\mu$ is σ-finite (by considering the sequence $([0,n])_{n\geq 1}$), the mapping theorem implies that $T\#M$ is a Poisson point process on $[0, \infty)$ with intensity measure $T\#\mu$.

Let $s \geq r \geq 0$. Then we have
$$T\#\mu([r,s]) = \mu\big(T^{-1}([r,s])\big) = \mu\big(B_s \setminus B_r\big) = \lambda \cdot (|B_s| - |B_r|) = \lambda \cdot \frac{\pi^{d/2}}{\Gamma(d/2+1)} \cdot (s^d - r^d).$$
More generally, $T\#\mu(B) = \lambda \cdot \mathrm{Leb}(T^{-1}(B))$ for $B \in \mathcal{B}([0, \infty))$.

(b) Fix a sequence $(r_k)_{k\geq 0}$ with $|B_{r_k}| = k$. Using the restriction property from Section 6.8, we note that $\big(M(B_{r_k} \setminus B_{r_{k-1}})\big)_{k\geq 1}$ is a sequence of independent, identically distributed random variables with
$$M(B_{r_1} \setminus B_{r_0}) \sim \mathrm{Pois}(\lambda).$$
Hence, by the strong law of large numbers, we have almost surely
$$\lim_{r\to\infty} \frac{M(B_r)}{|B_r|} = \lim_{n\to\infty} \frac{M(B_{r_n})}{|B_{r_n}|} = \lim_{n\to\infty} \frac{\sum_{k=1}^n M(B_{r_k} \setminus B_{r_{k-1}})}{n} = \mathbb{E}[M(B_{r_1} \setminus B_{r_0})] = \lambda.$$

Solution 11.3

(a) To see that the restricted measures are diffuse, it suffices to note that for every $i \in \mathbb{N}$,

$$\mu_i(\{x\}) \leq \mu(\{x\}) = 0, \quad \forall x \in E_i,$$

and so μ_i is diffuse.

The fact that $M_{E_1}, M_{E_2}, \ldots$ are independent Poisson point processes with respective intensities $\mu_{E_1}, \mu_{E_2}, \ldots$ follows directly from the restriction property in Section 6.8.

(b) Fix any $i \in \mathbb{N}$. Since $\mu(E_i) < \infty$, we can use Proposition 6.10 to the explicitly construct a Poisson point process with intensity measure μ_{E_i} as

$$\widetilde{M}_{E_i} = \sum_{j=1}^{Z} \delta_{X_j},$$

where $Z \sim \text{Pois}(\mu(E_i))$ and $X_j \sim \frac{\mu_{E_i}(\cdot)}{\mu_{E_i}}$, $j \geq 1$, are independent. Since being simple is a property of the law and $P_{\widetilde{M}_{E_i}} = P_{M_{E_i}}$ by Proposition 6.14, it suffices to prove that $\widetilde{M}_{E_i}$ is almost surely simple. To this end, we compute

$$\mathbb{P}[\widetilde{M}_{E_i} \text{ is not simple}] \leq \mathbb{P}[\exists j \neq k : X_j = X_k] \leq \sum_{j \neq k} \mathbb{P}[X_j = X_k] = 0,$$

where we used in the last equality that

$$\mathbb{P}[X_j = X_k] = \int_{E_i} \underbrace{\mathbb{P}[X_j = x]}_{=0} \frac{\mu_{E_i}(dx)}{\mu(E_i)}$$

by the independence of X_j and X_k. This concludes that $\widetilde{M}_{E_i}$ and thereby M_{E_i} is almost surely simple.

(c) Since $\mathbb{P}[M_{E_i} \text{ is simple}] = 1$ by part (b), we deduce that

$$\mathbb{P}[M \text{ is simple}] = \mathbb{P}[\bigcap_{i=1}^{\infty} \{M_{E_i} \text{ is simple}\}] = 1.$$

Solution 11.4

(a) Since the Lebesgue measure on $\mathbb{R} \times [0, \infty)$ is diffuse and σ-finite, M is almost surely simple.

(b) For $x \in \mathbb{Z} \subset \mathbb{R}$, $\mu(\{x\}) = 1$ and so $M(\{x\}) \sim \text{Pois}(1)$, which takes values in $\{2, 3, \ldots\}$ with positive probability. Hence, the process is *not* almost surely simple.

Solution 11.5
To show $(i) \iff (ii)$, we note that by definition,

$$P_M = P_{M'} \iff \forall A \in \mathcal{B}(\mathcal{M}), P_M(A) = P_{M'}(A) \iff \forall A \in \mathcal{B}(\mathcal{M}), \mathbb{P}[M \in A] = \mathbb{P}[M' \in A].$$

The implications $(ii) \implies (iii) \implies (iv)$ are clear by inclusion.

To show $(iii) \implies (ii)$, we use Dynkin's lemma. The family

$$\mathcal{B} := \{\{\eta : \eta(B_1) = n_1, \ldots, \eta(B_k) = n_k\} : k \geq 1; B_1, \ldots, B_k \in \mathcal{E} \text{ disjoint }; n_1, \ldots, n_k \in \mathbb{M}\} \subset \mathcal{B}(\mathcal{M})$$

is a π-system and $\sigma(\mathcal{B}) = \mathcal{B}(\mathcal{M})$ by definition. The family

$$\mathcal{D} := \{A \in \mathcal{B}(\mathcal{M}) : \mathbb{P}[M \in A] = \mathbb{P}[M' \in A]\}$$

is a Dynkin-system and it contains $\mathcal{B}$ by assumption. Hence, we conclude using Dynkin's lemma that $\mathcal{D} = \mathcal{B}(\mathcal{M})$ and so (ii) holds.

To show $(iv) \implies (iii)$, we consider $B_1, \ldots, B_k$ and $n_1, \ldots, n_k$ for some $k \geq 1$ and define the disjoint sets

$$C_1 = B_1,\ C_2 = B_2 \setminus B_1,\ \ldots,\ C_k = B_k \setminus \bigcup_{i=1}^{k-1} B_i.$$

(iii) then follows from (iv) by summing over all possible ways how the points could be distributed over the disjoint sets $C_1, \ldots, C_k$ under the constraints $M(B_1) = n_1, \ldots, M(B_k) = n_k$.