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Applied Stochastic Processes

Solution sheet 12

Solution 12.1

(a) First, we note that the product measure of two σ-finite measures taking values in $\mathbb{N} \cup \{\infty\}$ is itself a σ-finite measure taking values in $\mathbb{N} \cup \{\infty\}$. By Exercise 9.2 (a), showing that N is a point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ is equivalent to showing that for all $C \in \mathcal{E} \otimes \mathcal{F}$, $N(C)$ is a random variable. By applying Dynkin's lemma, it actually suffices to show that $N(C)$ is a random variable for sets of the form $C = A \times B$ with $A \in \mathcal{E}$ and $B \in \mathcal{F}$ (which form a π-system). But in this case, we have by the definition of the product measure $N(A \times B) = N(A) \cdot N'(B)$, which is a product of two random variables and thus a random variable.

(b) No, N is not a Poisson point process. To illustrate this, we consider the following example: Let $E = F = \mathbb{R}$ and $\mu = \nu = \text{Leb}$. Then

$$\big\{N([1,2]^2) = 1, N([1,2] \times [3,4]) = 0\big\} = \big\{N([1,2]) = 1, N'([1,2]) = 1, N'([3,4]) = 0\big\},$$

and so

$$\mathbb{P}\left[N([3,4]^2) = 0 | N([1,2]^2) = 1, N([1,2] \times [3,4]) = 0\right] = 1 \neq \mathbb{P}\left[N([3,4]^2) = 0\right],$$

which shows that N cannot be a Poisson point process as it contradicts the independence property on disjoint sets.

(c) Yes, this is exactly the marking theorem from Section 4.8.

(d) Yes, we can define a probability measure $\bar{\nu}$ by $\bar{\nu}(B) = \frac{\nu(B)}{\nu(F)}$ for $B \in \mathcal{F}$ and consider a Poisson point process M on E with intensity measure $\nu(F) \cdot \mu$. Then by the marking theorem, the marked process $\overline{M}$ is a Poisson point process on $E \times F$ with intensity measure $(\nu(F) \cdot \mu) \otimes \bar{\nu} = \mu \otimes \nu$. The equality of the two measures can be obtained by first noticing that they agree on sets of the form $C = A \times B$ for $A \in \mathcal{E}$ and $B \in \mathcal{F}$ and then applying Dynkin's lemma.

Solution 12.2

(a) Let us consider the marked process $\overline{M} = \sum_i \delta_{(X_i, R_i)}$. By the marking theorem, this is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}^+$ with intensity measure $\mu \otimes \rho$. In this process, each point (x, r) of the space corresponds to the ball $B(x, r)$. Note that the number of balls that intersect the origin is given by the number of points in the set $A = \{(x, r) \in \mathbb{R}^d \times \mathbb{R}^+ : |x| < r\}$. In other words $N_0 = \overline{M}(A)$. To see that A is measurable, we note that $A := f^{-1}((0, \infty))$ for the measurable function $f((x, r)) = r - |x|$. This implies that N_0 is a well defined random variable and that $N_0 \sim \text{Poisson}((\mu \otimes \rho)(A))$. We know by Fubini's Theorem that

$$(\mu \otimes \rho)(A) = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_A(y)(\mu \otimes \rho)(dy) = \int_{\mathbb{R}^d} \int_{|x|}^{\infty} \rho(dr)\mu(dx),$$

which shows what we wanted.

(b) First, note that $\{\mathcal{O} = \mathbb{R}^d\} = \bigcap_{N\geq 1}\{\mathcal{O} \supset \overline{B(0,N)}\}$. By compactness of $\overline{B(0,N)}$,

$$\{\mathcal{O} \supset \overline{B(0,N)}\} = \bigcup_{I\geq 1}\left\{\overline{B(0,N)} \subset \bigcup_{1\leq i\leq I} B(X_i,R_i)\right\},$$

and for every $I \in \mathbb{N}$,

$$\left\{\overline{B(0,N)} \subset \bigcup_{1\leq i\leq I} B(X_i,R_i)\right\} = \bigcup_{n\geq 1}\left\{\overline{B(0,N)} \subset \bigcup_{1\leq i\leq I} B(X_i,R_i) \text{ and } R_i \geq \frac{1}{n}, \forall i \leq I\right\}.$$

Since $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$, we have

$$\left\{\overline{B(0,N)} \subset \bigcup_{1\leq i\leq I} B(X_i,R_i) \text{ and } R_i \geq \frac{1}{n}, \forall i \leq I\right\} = \bigcap_{q\in\overline{B(0,N)}\cap\mathbb{Q}^d}\ \bigcup_{1\leq i\leq I} \underbrace{\{q \in B(X_i,R_i)\}}_{=\{|X_i-q|<R_i\}}.$$

In summary, we have shown that $\{\mathcal{O} = \mathbb{R}^d\}$ can be written in terms of countable unions and intersections of measurable sets, and is therefore measurable. Second, we know by Fubini's theorem that

$$\int_{\mathbb{R}^d}\int_{|x|}^{\infty} \rho(dr)\mu(dx) = \int_0^{\infty}\int_{B(0,r)} \mu(dx)\rho(dr) = \pi_d \int_0^{\infty} r^d \rho(dr)$$

Hence,

$$\mathbb{P}[0 \notin \mathcal{O}] = \mathbb{P}[N_0 = 0] = \exp\left(-\pi_d \int_0^{\infty} r^d\rho(dr)\right).$$

Suppose that $\mathbb{P}[\mathcal{O} = \mathbb{R}^d] = 1$. Then $P[0 \in \mathcal{O}] = 1$, and we deduce from the last expression that $\int_0^{\infty} r^d\rho(dr) = \infty$. To prove the converse, assume that $\int_0^{\infty} r^d\rho(dr) = \infty$. As a preliminary result we first show for any $n \in \mathbb{N}$ that

$$(\mu \otimes \rho)\left(\{(x,r) \in \mathbb{R}^d \times \mathbb{R}^+ : B(0,n) \subset B(x,r)\}\right) = \infty. \tag{1}$$

Since $B(0,n) \subset B(x,r)$ if and only if $r \geq |x| + n$, the left-hand side of equation (1) equals

$$\int_0^{\infty}\int_{\mathbb{R}^d} 1_{\{r\geq|x|+n\}}\mu(dx)\rho(dr) = \pi_d \int_n^{\infty} (r-n)^d\rho(dr).$$

This is bounded below by

$$\pi_d \int_{2n}^{\infty} \left(\frac{r}{2}\right)^d \rho(dr) = \pi_d 2^{-d}\int_0^{\infty} 1_{\{r\geq 2n\}} r^d\rho(dr),$$

proving (1). Since $\overline{M}$ is a Poisson point process with intensity $\mu \otimes \rho$, the ball $B(0,n)$ is almost surely covered even by one of the balls $B(X_i,R_i)$. Since n is arbitrary, it follows that $\mathbb{P}[\mathcal{O} = \mathbb{R}^d] = 1$.

Solution 12.3

(a) Recall by definition that $\mu(B) = \mathbb{E}[M(B)]$. We have

$$\mathcal{L}_M(t1_B) = \mathbb{E}\left[\exp\left(-t\int_E 1_B M(dx)\right)\right] = \mathbb{E}[\exp(-tM(B))].$$

Since $M(B) \geq 0$, the exponential above is bounded by 1. Besides, $M(B) \in L^1(\mathbb{P})$, so we can exchange the derivative and the expectation in the Laplace functional, therefore

$$-\frac{d}{dt}\mathcal{L}_M(t1_B) = \mathbb{E}[M(B)\exp(-tM(B))].$$

It suffices now to take $t = 0$ to conclude.

(b) For all $t > 0$, we have $L_M(t1_B) = \mathbb{E}[\exp(-tM(B))] = \mathbb{E}[1_{\{M(B)=0\}} + e^{-tM(B)}1_{\{M(B)\geq 1\}}]$. By dominated convergence, we get

$$\lim_{t\to\infty} \mathcal{L}_M(t1_B) = \mathbb{E}[1_{\{M(B)=0\}}] + 0 = \mathbb{P}[M(B) = 0].$$