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Applied Stochastic Processes

Solution sheet 13

Solution 13.1

(a) No. In order to be a Poisson process, the stochastic process $(N_t)_{t\geq 0}$ would also need to be almost surely non-decreasing and right-continuous (see the definition of counting process in Section 7.1).

(b) Let U be a uniform random variable taking values in $(0,1]$. Define $(N_t)_{t\geq 0}$ by

$$N_t := \sum_{i=0}^{\infty} \mathbb{1}_{i+U\leq t}.$$

It follows directly from the definition that $(N_t)_{t\geq 0}$ is a counting process and it makes jumps of size 1 at the times $U, 1+U, 2+U, \ldots$. The process has stationary increments since for any $t > s \geq 0$, the increment $N_t - N_s$ only depends on $t-s$. To show that the increments are *not* independent, it suffices to note that

$$N_{1/2} - N_0 = \mathbb{1}_{0<U\leq 1/2} \quad \text{and} \quad N_1 - N_{1/2} = \mathbb{1}_{1/2<U\leq 1} = 1 - (N_{1/2} - N_0).$$

(c) Yes. By definition, a counting process is almost surely non-decreasing and right-continuous. In particular, left limits almost surely exist due to the monotonicity of the process.

(d) Yes. By definition, a counting process almost surely is non-decreasing and takes values in $\mathbb{N}$. Hence, for any $t \geq 0$, the number of jumps in $[0,t]$ is at most N_t. Since the random variable N_t is almost surely finite, the same holds for the number of jumps.

Solution 13.2

(a) For the choice $\rho(u) = \lambda$ for all $u \geq 0$, we obtain for $0 \leq s < t$,

$$\int_s^t \rho(u)du = \lambda(t-s),$$

and so $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$. Hence, it follows from part (iii) of Theorem 7.2 or part (ii) of Theorem 7.3 that $(N_t)_{t\geq 0}$ is a Poisson process with rate λ.

(b) In general, the increments are *not* stationary. In part (a), we have seen that the increments are stationary if ρ is constant. Conversely, if ρ is not constant, we can choose $u, v \geq 0$ such that $\rho(u) > \rho(v)$. Then for some $h > 0$ sufficiently small,

$$\int_u^{u+h} \rho(u)du > \int_v^{v+h} \rho(v)dv,$$

and so the increments $N_{u+h} - N_u$ and $N_{v+h} - N_v$ do not have the same distribution.

(c) The intensity measure μ_ρ of the Poisson point process M is defined by

$$\mu_\rho(B) = \int_B \rho(u)du$$

for $B \in \mathcal{B}(\mathbb{R}_+)$.

(d) In general, S_1 and $S_2 - S_1$ are *not* independent as the following result shows.

Claim: S_1 and $S_2 - S_1$ are independent if and only if ρ is constant.

($\Longleftarrow$): If ρ is constant, then by part (a), $(N_t)_{t\geq 0}$ is actually a Poisson process with rate $\lambda = \rho$. Hence, the inter-arrival times are independent $\mathrm{Exp}(\lambda)$-distributed random variables.

($\Longrightarrow$): Let $s, t \geq 0$. For every $\epsilon \in (0, s)$, we have by the independence of the increments that

$$\mathbb{P}[t < S_1 \leq t + \epsilon] = \mathbb{P}[N_t = 0, N_{t+\epsilon} - N_t \geq 1] = \mathbb{P}[N_t = 0] \cdot \mathbb{P}[N_{t+\epsilon} - N_t \geq 1],$$

and

$$\begin{aligned}\mathbb{P}[S_2 - S_1 > s, t < S_1 \leq t + \epsilon] &= \mathbb{P}[N_t = 0, N_{t+\epsilon} - N_t = 1, N_{t+s} - N_{t+\epsilon} = 0] \\ &= \mathbb{P}[N_t = 0] \cdot \mathbb{P}[N_{t+\epsilon} - N_t = 1] \cdot \mathbb{P}[N_{t+s} - N_{t+\epsilon} = 0].\end{aligned}$$

By assumption, S_1 and $S_2 - S_1$ are independent and so we have

$$\mathbb{P}[S_2 - S_1 > s] = \mathbb{P}[S_2 - S_1 > s \,|\, t < S_1 \leq t + \epsilon] = \mathbb{P}[N_{t+s} - N_{t+\epsilon} = 0] \cdot \frac{\mathbb{P}[N_{t+\epsilon} - N_t = 1]}{\mathbb{P}[N_{t+\epsilon} - N_t \geq 1]}.$$

Letting $\epsilon \to 0$, it follows that

$$\mathbb{P}[S_2 - S_1 > s] = \exp\left(- \int_t^{t+s} \rho(u) du \right),$$

where we have used that $\frac{x \cdot e^{-x}}{1 - e^{-x}} \to 1$ as $x \to 0$. This is only possible if for all $s \geq 0$, $\int_t^{t+s} \rho(u) du$ does not depend on t. As in part (b), we conclude that ρ must be constant.

Solution 13.3

(a) The function $\rho : [0, +\infty) \to (0, +\infty)$ is continuous, hence integrable, and so R is well-defined and continuous as a function of t. Since ρ is strictly positive, R is strictly increasing as a function of t, hence injective. Finally, since $\int_0^\infty \rho(u)du = +\infty$, R is surjective.

(b) Since R is a continuous, increasing bijection by part (a), $R^{-1} : [0, +\infty) \to [0, +\infty)$ is a well-defined continuous, increasing bijection. In particular, $R^{-1}(0) = 0$. This implies that $(\widetilde{N}_t)_{t\geq 0}$ is a counting process. Furthermore, for any $k \geq 1$ and $0 = t_0 < t_1 < \ldots t_k$, it holds that $0 = R^{-1}(t_0) < R^{-1}(t_1) < \ldots < R^{-1}(t_k)$, and so the independence of the increments of $\widetilde{N}$ follows from the independence of the increments of N. Finally, for $0 \leq s < t$,

$$\widetilde{N}_t - \widetilde{N}_s = N_{R^{-1}(t)} - N_{R^{-1}(s)} \sim \mathrm{Pois}\Big(\underbrace{\int_{R^{-1}(s)}^{R^{-1}(t)} \rho(u)du}_{=t-s}\Big),$$

and so we conclude that $\widetilde{N}$ is a Poisson process with rate 1.

(c) As in part (c), we first note that $(N_t)_{t\geq 0}$ is a counting process with independent increments. Furthermore, for $0 \leq s < t$,

$$N_t - N_s = \widetilde{N}_{R(t)} - \widetilde{N}_{R(s)} \sim \mathrm{Pois}(\underbrace{R(t) - R(s)}_{\int_s^t \rho(u)du}),$$

and so we conclude that N is an inhomogeneous Poisson process with rate ρ.

Remark: Alternatively, it is possible to prove (b) and (c) using the mapping theorem for Poisson point processes from Section 6.9 and the correspondence between Poisson processes and Poisson point processes established in Theorem 7.2 (as well as an analogous result for inhomogenous Poisson processes).

Solution 13.4 Let $t > 0$ be fixed.

(a) Let us consider $0 \leq u \leq t$, $0 \leq v$. We have that

$$\mathbb{P}[A_t \geq u, B_t > v] = \mathbb{P}[S_{N_t} \leq t - u, S_{N_t+1} > t + v] = \mathbb{P}[N_{t+v} - N_{t-u} = 0] = e^{-\lambda u}e^{-\lambda v}$$

and if $u > t$ we have $\mathbb{P}[A_t \geq u, B_t > v] = 0$. Let U, V be independent random variables with distribution $\mathrm{Exp}(\lambda)$. Note that

$$\mathbb{P}[U \wedge t \geq u, V > v] = e^{-\lambda u}e^{-\lambda v}1_{\{u\leq t\}}.$$

Denote $\mu_{(A_t,B_t)}$ the joint law of A_t and B_t, and $\mu_{(U\wedge t,V)} = \mu_{U\wedge t} \otimes \mu_V$ the joint law of $U \wedge t$ and V, which is the product of their laws by independence. These measures agree on the set $\{[u, \infty) \times (v, \infty); u, v \in \mathbb{R}\}$, which is a π-system that generates $\mathcal{B}(\mathbb{R}^2)$. By Dynkin's Lemma, this implies that $\mu_{(A_t,B_t)} = \mu_{U\wedge t} \otimes \mu_V$. Therefore, for $u, v \geq 0$

$$\mathbb{P}[A_t \geq u] = \mu_{(A_t,B_t)}([u, \infty) \times \mathbb{R}) = \mu_{U\wedge t} \otimes \mu_V([u, \infty) \times \mathbb{R}) = e^{-\lambda u}1_{\{u\leq t\}}$$

and

$$\mathbb{P}[B_t > v] = \mu_{(A_t,B_t)}(\mathbb{R} \times (v, \infty)) = \mu_{U\wedge t} \otimes \mu_V(\mathbb{R} \times (v, \infty)) = e^{-\lambda v}.$$

This shows that $A_t \sim T_1 \wedge t$ and that $B_t \sim T_1$. We can also see from the steps above that for any $u, v \in \mathbb{R}$,

$$\mathbb{P}[A_t \geq u, B_t > v] = \mathbb{P}[A_t \geq u]\mathbb{P}[B_t > v].$$

Since the families $\{[u, \infty); u \in \mathbb{R}\}$ and $\{(v, \infty); v \in \mathbb{R}\}$ are π-systems that generate $\mathcal{B}(\mathbb{R})$, we conclude using Dynkin's Lemma that A_t and B_t are independent.

(b) From (a) we know that the densities of A_t and B_t are given by

$$f_{A_t}(x) = 1_{\{0 \leq x < t\}} \lambda e^{-\lambda x} + e^{-\lambda t} \delta_{(x,t)}, \quad f_{B_t}(x) = 1_{\{x \geq 0\}} \lambda e^{-\lambda x}.$$

Since A_t and B_t are independent, the density of L_t, is given by the convolution of f_{A_t} and f_{B_t}:

$$f_{L_t}(x) = \int_{\mathbb{R}} f_{A_t}(x-y) f_{B_t}(y) dy.$$

For $0 \leq x < t$:

$$f_{L_t}(x) = \int_0^x \lambda e^{-\lambda(x-y)} \lambda e^{-\lambda y} dy = \lambda^2 x e^{-\lambda x}.$$

For $x \geq t$:

$$f_{L_t}(x) = \int_{x-t}^x \lambda e^{-\lambda(x-y)} \lambda e^{-\lambda(y)} dy + e^{-\lambda t} \lambda e^{-\lambda(x-t)} = \lambda(1+\lambda t) e^{-\lambda x}$$

Hence,

$$\mathbb{E}[L_t] = \int_0^\infty x f_{L_t}(x) dx = \frac{2 - \exp(-\lambda t)}{\lambda}.$$

It follows that

$$\lim_{t \to \infty} \mathbb{E}[L_t] = \frac{2}{\lambda} = 2\mathbb{E}[T_1].$$

We discover that the interval in which t falls is not a "typical" interval. To give a short explanation note that the probability of $t > 0$ lying in a large interval is larger than the probability of t being contained in a short interval. This bias causes the selected interval to be on the average twice as long as a typical interval.

Remark: Alternatively, parts (a) and (b) can be established using the correspondence between Poisson processes and Poisson point processes established in Theorem 7.2.