Ritvik Radhakrishnan
Laurin Köhler-Schindler

Applied Stochastic Processes

Solution sheet 14

Solution 14.1

(a) Yes, $(N_t)_{t\geq 0}$ is a Poisson process with rate 3λ. Indeed, by superposition (Theorem 7.10), $(N_t^2 + N_t^3)_{t\geq 0}$ is a Poisson process with rate 2λ, independent of $(N_t^1)_{t\geq 0}$. Again applying superposition, we deduce that $(N_t)_{t\geq 0}$ is a Poisson process with rate $\lambda + 2\lambda = 3\lambda$.

(b) It follows from Theorem 7.10 that the probability that the k'th jump time of $(N_t)_{t\geq 0}$ is a jump time of $(N_t^1)_{t\geq 0}$ is equal to

$$\frac{\lambda}{\lambda + 2\lambda} = \frac{1}{3}.$$

(c) No, it is not a Poisson process since its jumps are of size 2.

Solution 14.2

(a) Let us denote φ_X the characteristic function of X_1. For every $s \in \mathbb{R}$ we have that

$$\begin{aligned}\varphi_{Z_t}(s) = \mathbb{E}[\exp(isZ_t)] = \mathbb{E}\left[\exp\left(is\sum_{k=1}^{N_t} X_k\right)\right] &= \mathbb{E}\left[\sum_{j=0}^{\infty}\exp\left(is\sum_{k=1}^{j} X_k\right)1_{\{N_t=j\}}\right] \\ &\overset{(1)}{=} \sum_{j=0}^{\infty}\mathbb{E}\left[\exp\left(is\sum_{k=1}^{j} X_k\right)\right]\cdot\mathbb{P}[N_t=j] \overset{(2)}{=} \sum_{j=0}^{\infty}\varphi_X(s)^j\cdot\frac{e^{-\lambda t}(\lambda t)^j}{j!} \\ &= \exp(\lambda t(\varphi_X(s)-1)).\end{aligned}$$

In (1) we used the dominated convergence theorem and independence between N_t and the X_i's. In (2) we used independence of the X_i's and that $N_t \sim \text{Pois}(\lambda t)$. We also used the convention that empty sums are equal to 0.

(b) Note that for every $n \geq 2$, $0 = t_0 < t_1 < \cdots < t_n < \infty$ and $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned}\mathbb{P}[Z_{t_1} - Z_{t_0} \in A_1, \ldots, Z_{t_n} - Z_{t_{n-1}} \in A_n] &= \mathbb{P}\left[\sum_{k=N_{t_0}+1}^{N_{t_1}} X_k \in A_1, \ldots, \sum_{k=N_{t_{n-1}}+1}^{N_{t_n}} X_k \in A_n\right] \\ = \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n}\mathbb{P}\left[\sum_{k=1}^{i_1} X_k \in A_1, \ldots, \sum_{k=i_1+\cdots+i_{n-1}+1}^{i_1+\cdots i_n} X_k \in A_n\right]&\mathbb{P}[N_{t_1} = i_1, \ldots, N_{t_n} - N_{t_{n-1}} = i_n],\end{aligned} \tag{1}$$

where we used that $(N_t)_{t\geq 0}$ is independent of the X_i's. Since the increments of a Poisson process are stationary, we have that for $h > 0$

$$\mathbb{P}[N_{t_1} = i_1, \ldots, N_{t_n} - N_{t_{n-1}} = i_n] = \mathbb{P}[N_{t_1+h} - N_h = i_1, \ldots, N_{t_n+h} - N_{t_{n-1}+h} = i_n].$$

Then, replacing this in (1), and coming back through the same steps, we obtain

$$\mathbb{P}[Z_{t_1} - Z_{t_0} \in A_1, \ldots, Z_{t_n} - Z_{t_{n-1}} \in A_n] = \mathbb{P}[Z_{t_1+h} - Z_{t_0+h} \in A_1, \ldots, Z_{t_n+h} - Z_{t_{n-1}+h} \in A_n]$$

i.e., $(Z_{t_1} - Z_{t_0}, \dots, Z_{t_n} - Z_{t_{n-1}}) \overset{(d)}{=} (Z_{t_1+h} - Z_{t_0+h}, \dots, Z_{t_n+h} - Z_{t_{n-1}+h})$, and the process Z has stationary increments. If now we use the fact that the increments of the Poisson process $(N_t)_{t\geq 0}$ are independent, and that the random variables X_i's are also independent, we have that (1) equals to

$$\prod_{j=1}^{n} \sum_{i_j=1}^{\infty} \mathbb{P}\left[\sum_{k=i_1+\dots+i_{j-1}+1}^{i_1+\dots+i_j} X_k \in A_j\right] \mathbb{P}[N_{t_j} - N_{t_{j-1}} = i_j] = \prod_{j=1}^{n} \mathbb{P}\left[\sum_{k=N_{t_{j-1}}+1}^{N_{t_j}} X_k \in A_j\right]$$
$$= \prod_{j=1}^{n} \mathbb{P}[Z_{t_j} - Z_{t_{j-1}} \in A_j],$$

which means that Z has independent increments.

(c) **Option 1:** First, since $X_k \in \{0,1\}$, $k \geq 1$, $\mathbb{P}$-a.s., it follows that Z is a counting process. Second, since $X_i \sim \mathrm{Ber}(p)$, we have that $\varphi_X(s) = 1 + p(e^{is} - 1)$. Hence, using part (a), we have that $\varphi_{Z_t}(s) = \exp(\lambda pt(e^{is} - 1))$ and therefore $Z_t \sim \mathrm{Pois}(\lambda pt)$. Since it also has independent and stationary increments by part (b), it follows that it has the same finite marginals as a Poisson process with rate λp, which concludes using part (iii) of Theorem 7.2.

Option 2: We note that in this case,

$$Z_t = \sum_{k=1}^{N_t} X_k = \sum_{k=1}^{N_t} \mathbf{1}_{X_k=1} = \sum_{k\geq 1} \mathbf{1}_{S_k \leq t, X_k=1}.$$

It now follows from thinning (Theorem 7.8) that $(Z_t)_{t\geq 0}$ is a Poisson process with rate λp.

Solution 14.3

(a) First we will show that almost surely there exists n_0 such that for all $n \geq n_0$ we have

$$T_n \leq \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).$$

Set $E_n := \{T_n > \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda)\}$, then

$$\mathbb{P}[E_n] = \exp\left(-\lambda \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda)\right) = \left(\frac{\lambda}{n}\right)^{1+\varepsilon},$$

hence $\sum_n \mathbb{P}[E_n] < \infty$ and therefore by Borel-Cantelli, we obtain $\mathbb{P}[\limsup_{n\to\infty} E_n] = 0$. This means that for almost every ω, there is $n_0(\omega)$ such that for all $n \geq n_0(\omega)$ we have

$$\max_{n_0(\omega)\leq k\leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \max_{n_0(\omega)\leq k\leq n} \log(k/\lambda) = \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).$$

Furthermore, we can choose $n_1(\omega) \geq n_0(\omega)$ such that

$$\max_{1\leq k\leq n_0(\omega)} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \log(n_1(\omega)/\lambda),$$

because log is a monotone function increasing to infinity. Therefore almost surely, there is n_1 such that for all $n \geq n_1$, we have

$$\max_{1\leq k\leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).$$

(b) We have $\limsup_{t\to\infty} \frac{N_t+1}{t} = \limsup_{t\to\infty} \frac{N_t}{t}$ and

$$\limsup_{t\to\infty} \frac{N_t}{t} \leq \limsup_{t\to\infty} \frac{N_t}{S_{N_t}} = \limsup_{k\to\infty} \frac{k}{S_k} = \lambda,$$

where we used in the last step that by the strong law of large numbers we have $S_k/k \to \frac{1}{\lambda}$ almost surely as $k \to \infty$. This implies that almost surely there is t_0 such that for all $t > t_0$ we have

$$\frac{N_t + 1}{t} \leq (1+\varepsilon)\lambda.$$

(c) Almost surely for t large enough we have

$$L_t \leq \max_{1\leq k\leq N_t+1} T_k \leq \frac{(1+\varepsilon)}{\lambda} \log\left(\frac{N_t+1}{\lambda}\right) \leq \frac{(1+\varepsilon)}{\lambda} \log(t(1+\varepsilon)),$$

which yields $\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \frac{(1+\varepsilon)}{\lambda}$. As $\varepsilon > 0$ was arbitrarily chosen this yields the claim.