Ritvik Radhakrishnan
Laurin Köhler-Schindler

Applied Stochastic Processes

Solution sheet 2

Solution 2.1

(a) Under the measure $\mathbf{P}_x$, we have $X_0 \sim \delta^x$. We use Definition 1.3 to obtain

$$\mathbf{P}_x[X_1 = y, X_0 = x] = \delta^x(x) \cdot p_{xy} = p_{xy},$$
$$\mathbf{P}_x[X_1 = x, X_0 = y] = \delta^x(y) \cdot p_{yx} = \begin{cases} 0 & \text{if } x \neq y, \\ p_{xx} & \text{if } x = y. \end{cases}$$

(b) We apply the simple Markov property with $k = n$, $Z = 1$ and $f((X_{k+m})_{m\geq 0}) = \mathbb{1}_{X_{k+2}=z, X_{k+1}=y}$ to obtain

$$\mathbf{P}_x[X_{n+2} = z, X_{n+1} = y | X_n = x] = \mathbf{E}_x[\mathbb{1}_{X_2=z, X_1=y}] = \mathbf{P}_x[X_2 = z, X_1 = y] = p_{xy} \cdot p_{yz}.$$

By definition of the n-step transition probability, we obtain

$$\begin{aligned}\mathbf{P}_x[X_{n+2} = z, X_{n+1} = y, X_n = x] &= p_{xx}^{(n)} \cdot p_{xy} \cdot p_{yz} \\ &= \sum_{x_1,\dots,x_{n-1}\in S} p_{xx_1} \cdot \ldots p_{x_{n-1}x} \cdot p_{xy} \cdot p_{yz}.\end{aligned}$$

(c)

$$\mathbf{P}_1[X_1 = 3] = p_{13} = 0 \qquad \mathbf{P}_1[X_2 = 3] = p_{12} \cdot p_{23} = 1/4$$
$$\mathbf{P}_1[X_3 = 3] = 0 \qquad \mathbf{P}_1[X_4 = 3] = \binom{4}{1} \cdot 1/16 = 1/4.$$

In the third case, we used that there exits no nearest-neighbor walk on $\mathbb{Z}$ from 1 to 3 of length 3. In the fourth case, we used that every nearest-neighbor path on $\mathbb{Z}$ from 1 to 3 of length 4 does exactly 3 steps "+1" and 1 step "-1". Each such path has probability 1/16 and there are $\binom{4}{1}$ ways to choose the position of the step "-1".

(d) First, we note that there exists no nearest-neighbor walk on $\mathbb{Z}$ from 0 to 0 of odd length. Hence, for n odd, we obtain $\mathbf{P}_0[X_n = 0] = 0$. Second, for n even, a nearest-neighbor walk on $\mathbb{Z}$ from 0 to 0 does exactly $n/2$ steps "+1" and $n/2$ steps "-1". Hence, there are $\binom{n}{n/2}$ ways to choose the positions of the steps "-1", and we obtain

$$\mathbf{P}_0[X_n = 0] = \binom{n}{n/2} 2^{-n}.$$

Solution 2.2

(a) By Chapman-Kolmogorov (Proposition 1.8), we have

$$p_{0x}^{(2n)} = \sum_{y\in S} p_{0y}^{(n)} \cdot p_{yx}^{(n)}.$$

Using the Chauchy-Schwartz inequality, we obtain

$$p_{0x}^{(2n)} = \sum_{y\in S} p_{0y}^{(n)} \cdot p_{yx}^{(n)} \leq \sqrt{\left(\sum_{y\in S}\left(p_{0y}^{(n)}\right)^2\right) \cdot \left(\sum_{y\in S}\left(p_{yx}^{(n)}\right)^2\right)}$$

(b) Since the transition probability of the SRW is symmetric with respect to permuting x and y, i.e. $p_{xy} = p_{yx}$, we obtain

$$\sum_{y\in S}\left(p_{0y}^{(n)}\right)^2 = \sum_{y\in S} p_{0y}^{(n)} \cdot p_{y0}^{(n)} = p_{00}^{(2n)},$$

where we again used Chapman-Kolmogorov. Analogously,

$$\sum_{y\in S}\left(p_{yx}^{(n)}\right)^2 = \sum_{y\in S} p_{xy}^{(n)} \cdot p_{yx}^{(n)} = p_{xx}^{(2n)}.$$

Combining all previous steps, we obtain

$$p_{0x}^{(2n)} \le \sqrt{\left(\sum_{y\in S}\left(p_{0y}^{(n)}\right)^2\right)\cdot\left(\sum_{y\in S}\left(p_{yx}^{(n)}\right)^2\right)} = \sqrt{p_{00}^{(2n)}\cdot p_{xx}^{(2n)}} = p_{00}^{(2n)},$$

where we used $p_{00}^{(2n)} = p_{xx}^{(2n)}$ in the last step.

Solution 2.3

Under $\mathbf{P}_0$, $(X_n)_{n\ge 0}$ is a simple random walk (SRW) starting at 0. For $i \in \mathbb{Z}$ and $k \ge 0$,

$$\begin{aligned}\mathbf{P}_0\left[Z' = k | X_{10} = i\right] &= \mathbf{P}_0\left[\left(\sum_{n=10}^{20} \mathbb{1}_{X_n=i}\right) = k | X_{10} = i\right] = \mathbf{P}_i\left[\left(\sum_{n=0}^{10} \mathbb{1}_{X_n=i}\right) = k\right] \\ &= \mathbf{P}_0\left[\left(\sum_{n=0}^{10} \mathbb{1}_{X_n=0}\right) = k\right] = \mathbf{P}_0\left[Z = k\right]\end{aligned}$$

where the second equality follows from the simple Markov property and the third equality follows since $(i + X_n)_{n\ge 0}$ is a SRW starting at i (under $\mathbf{P}_0$). Since the right-hand side does not depend on i, it directly follows that

$$\mathbf{P}_0[Z' = k] = \sum_{i\in\mathbb{Z}} \mathbf{P}_0\left[Z' = k | X_{10} = i\right] \cdot \mathbf{P}_0\left[X_{10} = i\right] = \mathbf{P}_0\left[Z = k\right],$$

and so Z and Z' have the same distribution. Furthermore, we see that Z' and X_{10} are independent.The Markov property directly implies that Z and Z' are conditionally independent given $\{X_{10} = i\}$. Therefore, Z and Z' are independent as the following computation shows:

$$\begin{aligned}\mathbf{P}_0[Z = k, Z' = \ell] &= \sum_{i\in\mathbb{Z}} \mathbf{P}_0[Z = k, Z' = \ell | X_{10} = i] \cdot \mathbf{P}_0[X_{10} = i] \\ &= \sum_{i\in\mathbb{Z}} \mathbf{P}_0[Z = k | X_{10} = i] \cdot \mathbf{P}_0[Z' = \ell | X_{10} = i] \cdot \mathbf{P}_0[X_{10} = i] \\ &= \mathbf{P}_0[Z' = \ell] \cdot \left(\sum_{i\in\mathbb{Z}} \mathbf{P}_0[Z = k | X_{10} = i] \cdot \mathbf{P}_0[X_{10} = i]\right) \\ &= \mathbf{P}_0[Z' = \ell] \cdot \mathbf{P}_0[Z = k].\end{aligned}$$

Solution 2.4

(a) We establish the inequality by induction on k. For $k=0$, the inequality is trivial. For $k \geq 1$, it follows from the simple Markov property that

$$\begin{aligned}
&\mathbf{P}_0[H_{-N,N} > k \cdot N] \\
&= \sum_{-N+1 \leq x_1,\dots,x_{(k-1)N} \leq N-1} \mathbf{P}_0[H_{-N,N} > k \cdot N, X_1 = x_1, \dots, X_{(k-1)N} = x_{(k-1)N}] \\
&= \sum_{-N+1 \leq x_1,\dots,x_{(k-1)N} \leq N-1} \mathbf{P}_{x_{(k-1)N}}[H_{-N,N} > N] \cdot \mathbf{P}_0[X_1 = x_1, \dots, X_{(k-1)N} = x_{(k-1)N}]
\end{aligned}$$

Since the distance from any $x \in \{-N+1, \dots, N-1\}$ to either N or $-N$ is at most N, it follows that $\mathbf{P}_x[H_{-N,N} \leq N] \geq 2^{-N}$. Thus,

$$\begin{aligned}
&\mathbf{P}_0[H_{-N,N} > k \cdot N] \\
&\leq (1-2^{-N}) \cdot \sum_{-N+1 \leq x_1,\dots,x_{(k-1)N} \leq N-1} \mathbf{P}_0[X_1 = x_1, \dots, X_{(k-1)N} = x_{(k-1)N}] \\
&= (1-2^{-N}) \cdot \mathbf{P}_0[H_{-N,N} > (k-1) \cdot N] \leq (1-2^{-N})^k,
\end{aligned}$$

where we used the induction hypothesis in the last step.

We compute

$$\mathbf{E}_0[H_{-N,N}] = \sum_{\ell=0}^{\infty} \mathbf{P}_0[H_{-N,N} > \ell] \leq \sum_{k=0}^{\infty} N \cdot \underbrace{\mathbf{P}_0[H_{-N,N} > k \cdot N]}_{(1-2^{-N})^k} = N \cdot 2^N.$$

(b) Assume towards a contradiction that $\mathbf{E}_x[H_{-N,N}] = \infty$ for some $x \in \{-N, \dots, N\}$. Without loss of generality, let us assume that x is a non-negative integer. Then $p_{0x}^{(x)} = 2^{-x}$, and so by the simple Markov property

$$\begin{aligned}
\mathbf{E}_0[H_{-N,N}] &\geq \mathbf{E}_0[H_{-N,N} \cdot \mathbb{1}_{X_1=1,\dots,X_x=x}] \\
&= \underbrace{\mathbf{E}_x[(H_{-N,N} + x)]}_{=\infty} \cdot \underbrace{\mathbf{P}_0[X_1 = 1, \dots, X_x = x]}_{=2^{-x}} = \infty,
\end{aligned}$$

which contradicts the result of (a).

(c) First, we note that by (b), the function $f : \{-N, \dots, N\} \to \mathbb{R}_+$, given by

$$f(x) = \mathbf{E}_x[H_{-N,N}],$$

is well-defined. Moreover, f is even (i.e. $f(x) = f(-x)$) due to the symmetry of the SRW, and it has boundary values $f(-N) = f(N) = 0$. For $x \in \{-N+1, \dots, N-1\}$,

$$\begin{aligned}
f(x) = \mathbf{E}_x[H_{-N,N}] &= \mathbf{E}_x[H_{-N,N} \cdot \mathbb{1}_{X_1=x-1}] + \mathbf{E}_x[H_{-N,N} \cdot \mathbb{1}_{X_1=x+1}] \\
&= \mathbf{E}_{x-1}[H_{-N,N} + 1] \cdot \mathbf{P}_x[X_1 = x-1] + \mathbf{E}_{x+1}[H_{-N,N} + 1] \cdot \mathbf{P}_x[X_1 = x+1] \\
&= (f(x-1)+1) \cdot \frac{1}{2} + (f(x+1)+1) \cdot \frac{1}{2} = \frac{f(x-1) + f(x+1)}{2} + 1
\end{aligned}$$

Equivalently, for every $x \in \{-N+1, \dots, N-1\}$,

$$f(x) - f(x-1) = f(x+1) - f(x) + 2.$$

Let $n \geq 0$. Summing over all $x \in \{-n, \dots, n\}$, it follows that

$$\begin{aligned} f(n) - f(-n-1) &= \sum_{x=-n}^{n} (f(x) - f(x-1)) \\ &= \sum_{x=-n}^{n} (f(x+1) - f(x) + 2) = f(n+1) - f(-n) + 2(2n+1). \end{aligned}$$

Thus, since f is even, we obtain

$$f(n) = f(n+1) + (2n+1).$$

Using $f(N) = 0$, we inductively obtain

$$\begin{aligned} f(n) &= \sum_{m=n}^{N-1} (2m+1) = 2 \cdot \left(\sum_{m=n}^{N-1} m \right) + (N-n) \\ &= 2 \cdot \left(\frac{N(N-1)}{2} - \frac{n(n-1}{2} \right) + (N-n) \\ &= N^2 - n^2. \end{aligned}$$

In particular, $f(0) = N^2$, which is what we wanted to show.

Remark: Another strategy would be to show that the function $g : \{-N, \dots, N\} \to \mathbb{R}_+$, defined by

$$g(x) = f(x) + x^2,$$

is harmonic in the interior of $\{-N, \dots, N\}$ and satisfies $g(-N) = g(N) = N^2$. Using the uniqueness of the solution to the Dirichlet problem, i.e. the fact that there is a unique harmonic function $h : \{-N, \dots, N\} \to \mathbb{R}_+$ satisfying the boundary condition $h(-N) = h(N) = N^2$, it then follows that $g(x) = N^2$ for every $x \in \{-N, \dots, N\}$. Thus, $f(x) = N^2 - x^2$ for every $x \in \{-N, \dots, N\}$.

Solution 2.5

(a) For E is finite, we choose the uniform measure, i.e. $\mu(x) := |E|^{-1}$ for every $x \in E$. For E countably infinite, we assume without loss of generality that $E = \{1, 2, 3, \ldots\}$. We note that

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

and choose $\mu(n) := 6/(n\pi)^2$ for every $n \geq 1$.

(b) Define μ to be the law of X_0 and set

$$p_{xy} = \begin{cases} \mathbb{P}[X_{n+1} = y | X_n = x] & \text{if } \exists n : \mathbb{P}[X_n = x] > 0, \\ \mathbb{1}_{x=y} & \text{otherwise.} \end{cases}$$

By homogeneity, p_{xy} is well-defined. Furthermore, for every $x_0, \ldots, x_n \in S$, we have

$$\begin{aligned} \mathbb{P}[X_0 = x_0, \ldots, X_n = x_n] &= \underbrace{\mathbb{P}[X_0 = x_0]}_{=\mu(x_0)} \cdot \prod_{i=1}^{n} \underbrace{\mathbb{P}[X_i = x_i | X_0 = x_0, \ldots, X_{i-1} = x_{i-1}]}_{=\mathbb{P}[X_i = x_i | X_{i-1} = x_{i-1}] = p_{x_{i-1}x_i}} \\ &= \mu(x_0) \cdot p_{x_0 x_1} \cdot \ldots \cdot p_{x_{n-1}x_n}, \end{aligned}$$

where we used the 1-step Markov property and the definitions of μ and P.

It remains to check that P is a transition probability. Let $x \in S$. If there exists $n \geq 0$ such that $\mathbb{P}[X_n = x] > 0$, then

$$\sum_{y \in E} p_{xy} = \sum_{y \in E} \mathbb{P}[X_{n+1} = y | X_n = x] = 1.$$

Otherwise,

$$\sum_{y \in E} p_{xy} = \sum_{y \in E} \mathbb{1}_{x=y} = 1.$$

Solution 2.6

(a) Based on the definition of $\mathcal{F}_T$ and T being an $(\mathcal{F}_n)$-stopping time, we obtain the following chain of equivalences:

$$\begin{aligned} & Z \text{ is } \mathcal{F}_T\text{-measurable} \\ \iff & \forall a \in \mathbb{R},\ \{Z \leq a\} \in \mathcal{F}_T \\ \iff & \forall a \in \mathbb{R}, \forall n \in \mathbb{N},\ \{Z \leq a\} \cap \{T = n\} \in \mathcal{F}_n \\ \iff & \forall n \in \mathbb{N},\ Z \cdot \mathbb{1}_{T=n} \text{ is } \mathcal{F}_n\text{-measurable} \end{aligned}$$

Note that for $a \geq 0$, $\{Z \cdot \mathbb{1}_{T=n} \leq a\} = (\{Z \leq a\} \cap \{T = n\}) \cup \{T = n\}^{\mathsf{c}}$. To establish the last equivalence, it then suffices to note that $\{T = n\} \in \mathcal{F}_n$ by definition of an $(\mathcal{F}_n)$-stopping time.

(b) Using the strong Markov property and $\mathrm{P}_\mu[T < \infty] = 1$, we obtain

$$\begin{aligned} \mathrm{E}_\mu\left[f\left((X_{T+n})_{n\geq 0}\right) \cdot Z | X_T = x\right] &= \frac{\mathrm{E}_\mu\left[f\left((X_{T+n})_{n\geq 0}\right) \cdot Z \cdot \mathbb{1}_{X_T = x}\right]}{\mathrm{P}_\mu[X_T = x]} \\ &= \frac{\mathrm{E}_\mu\left[f\left((X_{T+n})_{n\geq 0}\right) \cdot Z \cdot \mathbb{1}_{T<\infty, X_T = x}\right]}{\mathrm{P}_\mu[T < \infty, X_T = x]} \\ &= \mathrm{E}_\mu\left[f\left((X_{T+n})_{n\geq 0}\right) \cdot Z | T < \infty, X_T = x\right] \\ &= \mathrm{E}_x\left[f\left((X_n)_{n\geq 0}\right)\right] \cdot \mathrm{E}_\mu[Z | T < \infty, X_T = x] \\ &= \mathrm{E}_x\left[f\left((X_n)_{n\geq 0}\right)\right] \cdot \mathrm{E}_\mu\left[Z | X_T = x\right]. \end{aligned}$$

Solution 2.7 Let $x \in C$. We have $\mathbf{P}_x[\tau_C > kN] = \mathbf{P}_x[0 > kN] = 0$, and so the inequality holds true for all $k \geq 0$.

Let $x \in S \setminus C$. The inequality is trivial for $k = 0$. For $k \geq 1$, we prove it by induction over k. For $k = 1$, we have

$$\mathbf{P}_x[\tau_C > N] \leq \mathbf{P}_x[\tau_C > n(x)] = 1 - \mathbf{P}_x[\tau_C \leq n(x)] = 1 - \mathbf{P}_x[X_{n(x)} \in C] \leq 1 - \varepsilon. \tag{1}$$

For $k \geq 2$, it follows from the Markov property that

$$\begin{aligned}\mathbf{P}_x[\tau_C > kN] &= \sum_{y_1,\ldots,y_{(k-1)N} \in S \setminus C} \mathbf{P}_x[\tau_C > kN, X_1 = y_1, \ldots, X_{(k-1)N} = y_{(k-1)N}] \\ &= \sum_{y_1,\ldots,y_{(k-1)N} \in S \setminus C} \mathbf{P}_{y_{(k-1)N}}[\tau_C > N] \cdot \mathbf{P}[X_1 = y_1, \ldots, X_{(k-1)N} = y_{(k-1)N}].\end{aligned}$$

By (1), we have $P_{y_{(k-1)N}}[\tau_C > N] \leq 1 - \varepsilon$ for all $y_{(k-1)N} \in S$, and so

$$\begin{aligned}\mathbf{P}_x[\tau_C > kN] &\leq (1-\varepsilon) \cdot \sum_{y_1,\ldots,y_{(k-1)N} \in S \setminus C} \mathbf{P}_x[X_1 = y_1, \ldots, X_{(k-1)N} = y_{(k-1)N}] \\ &= (1-\varepsilon) \cdot \mathbf{P}_x[\tau_C > (k-1)N] \\ &\leq (1-\varepsilon)^k,\end{aligned}$$

where we used the induction hypothesis $\mathbf{P}_x[\tau_C > (k-1)N] \leq (1-\varepsilon)^{k-1}$ in the last equation.