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Applied Stochastic Processes

Solution sheet 3

Solution 3.1

(a) For $x \neq y$, we note that $\widetilde{H_x} \geq 1$ $\mathbf{P}_y$-a.s. since $X_0 = y \neq x$. Thus, $\widetilde{H_x} = H_x$ $\mathbf{P}_y$-a.s. and in particular,

$$\mathbf{P}_y[\widetilde{H_x} < \infty] = \mathbf{P}_y[H_x < \infty].$$

For $x = y$, we note that $\widetilde{H_x} = 0$ $\mathbf{P}_y$-a.s. since $X_0 = y = x$. Thus, $\mathbf{P}_y[\widetilde{H_x} < \infty] = 1$, but we cannot say anything about $\mathbf{P}_y[H_x < \infty]$.

(b) For $x \neq y$, we note that $X_0 = y \neq x$ $\mathbf{P}_y$-a.s. and thus,

$$\mathbf{E}_y[\widetilde{V_x}] = \mathbf{E}_y[V_x].$$

For $x = y$, we note that $X_0 = y = x$ $\mathbf{P}_y$-a.s. and thus,

$$\mathbf{E}_y[\widetilde{V_x}] = 1 + \mathbf{E}_y[V_x].$$

(c) The statement follows from the strong Markov property. More precisely, we will apply the strong Markov property for the stopping time $T = H_x$, the $\mathcal{F}_T$-measurable and bounded random variable $Z = 1$, and the measurable and bounded function $f_N : S^{\mathbb{N}} \to \mathbb{R}$ defined by

$$f_N(x_0, x_1, x_2, \dots) = \left(\sum_{n=0}^{\infty} \mathbf{1}_{x_n = x} \right) \wedge N,$$

where we needed to introduce the truncation at N to ensure that f_N is bounded. Since $X_{H_x} = x$ and $H_x = \infty \implies V_x = 0$ $\mathbf{P}_y$-almost-surely, we obtain

$$\begin{aligned} \mathbf{E}_y[V_x \wedge N] &= \mathbf{E}_y\left[(V_x \wedge N) \cdot \mathbf{1}_{H_x < \infty}\right] \\ &= \mathbf{E}_y\left[V_x \wedge N | H_x < \infty, X_{H_x} = x\right] \cdot \mathbf{P}_y\left[H_x < \infty\right]. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbf{E}_y\left[V_x \wedge N | H_x < \infty, X_{H_x} = x\right] &= \mathbf{E}_y\left[\left(\sum_{k \geq H_x} \mathbf{1}_{X_k = x} \right) \wedge N | H_x < \infty, X_{H_x} = x \right] \\ &= \mathbf{E}_y\left[f_N\left((X_{H_x+n})_{n\geq 0}\right) | H_x < \infty, X_{H_x} = x\right] \\ &= \mathbf{E}_x\left[f_N\left((X_n)_{n \geq 0}\right)\right] \\ &= \mathbf{E}_x\left[\widetilde{V_x} \wedge N\right], \end{aligned}$$

where we used the strong Markov property in the third equality. Taking the limit as $N \to \infty$, we conclude by monotone convergence that

$$\begin{aligned} \mathbf{E}_y[V_x] = \lim_{N\to\infty} \mathbf{E}_y[V_x \wedge N] &= \lim_{N \to \infty} \mathbf{E}_x\left[\widetilde{V_x} \wedge N\right] \cdot \mathbf{P}_y\left[H_x < \infty\right] \\ &= E_x\left[\widetilde{V_x}\right] \cdot \mathbf{P}_y\left[H_x < \infty\right] = (1 + \mathbf{E}_x[V_x]) \cdot \mathbf{P}_y\left[H_x < \infty\right], \end{aligned}$$

where we used (b) in the last equality.

(d) We have $V_x^{(n)} \leq V_x^{(n+1)}$, hence $\mathbf{E}_y[V_x^{(n)}] \leq \mathbf{E}_y[V_x^{(n+1)}]$.

(e) By monotone convergence, $\lim_{n\to\infty} \mathbf{E}_y[V_x^{(n)}] = \mathbf{E}_y[V_x]$.

(f) No, it does not hold for fixed $n \in \mathbb{N}$.
Counterexample: For $N \geq n$, consider the deterministic Markov chain with state space $\{0, 1, \dots, N\}$ and transition probability given by $p_{i,i+1} = 1$ for $0 \leq i < N$ and $p_{N,N} = 1$. For $y = 0$ and $x = N$, we obtain

$$\mathbf{E}_y[V_x^{(n)}] = 0 \neq (n+1) = \underbrace{\mathbf{P}_y[H_x < \infty]}_{=1} \cdot (1 + \underbrace{\mathbf{E}_x[V_x^{(n)}]}_{=n}).$$

(g) No, this would imply $\mathbf{E}_x[V_x] = \infty$ but $\mathbf{P}_x[V_x = \infty] < 1$, which contradicts the Dichotomy Theorem.

(h) For x recurrent, we have $\mathbf{P}_x[V_x = \infty] = 1$, so in order to have $\mathbf{P}_x[V_x = 2] > 0$, we need x to be transient. By Lemma 2.3,

$$\mathbf{P}_x[V_x = 2] = \mathbf{P}_x[V_x \geq 2] - \mathbf{P}_x[V_x \geq 3] = \rho_x^2 - \rho_x^3.$$

It is easy to verify that the function $f(\rho) = \rho^2 - \rho^3$ achieves its maximum on $[0, 1]$ at $\rho = 2/3$ with $f(2/3) = 4/27$. Consequently, it is possible to construct Markov chains with $\mathbf{P}_x[V_x = 2] = \rho$ if and only if $\rho \in [0, 4/27]$.

Concrete example for $\rho = 1/8$: We can consider the two-state Markov chain with state space $S = \{x, y\}$ and transition probability given by $p_{xx} = p_{xy} = 1/2$ and $p_{yy} = 1$. Then $\mathbf{P}_x[V_x = 2] = \mathbf{P}_x[X_1 = x, X_2 = x, X_3 = y] = p_{xx} \cdot p_{xx} \cdot p_{xy} = 1/8$.

(i) As argued in (h), this is not possible.

(j) If x is transient, then $\mathbf{P}_x[V_x = \infty] = 0$, and so $\mathbf{P}_x$-almost surely, there exists some (random) N such that for all $n \geq N$, $T_n = \infty$. Hence, $\mathbf{P}_x$-almost surely,

$$\lim_{n\to\infty} \frac{T_5 + \ldots + T_n}{n} = \infty.$$

If x is positive recurrent, then $\mathbf{E}_x[T_i] = \mathbf{E}[H_x] = m_x < \infty$, and so by the law of large numbers, $\mathbf{P}_x$-almost surely,

$$\lim_{n\to\infty} \frac{T_5 + \ldots + T_n}{n} = m_x.$$

If x is null recurrent, then for any $C < \infty$, there is K such that $\mathbf{E}_x[(T_i \wedge K] \geq C$. By the law of large numbers, $\mathbf{P}_x$-almost surely,

$$\lim_{n\to\infty} \frac{T_5 + \ldots + T_n}{n} \geq \lim_{n\to\infty} \frac{(T_5 \wedge K) + \ldots + (T_n \wedge K)}{n} \geq C.$$

Since C can be chosen arbitrarily large, this implies

$$\lim_{n\to\infty} \frac{T_5 + \ldots + T_n}{n} = \infty.$$

(k) By Remark 2.5 applied with $x = y$, the limit is equal to $\frac{1}{m_x}$.

Solution 3.2

(a) We prove recurrence of the state 0, but the same proof applies to all $x \in \mathbb{Z}$ (due to translation invariance of the transition probability of the SRW).
In Exercise 2.2 (b), it was shown that for the SRW on $\mathbb{Z}$, i.e. the case of $\alpha = 1/2$ in the context of this exercise, we have for every $x \in \mathbb{Z}$ and for every $n \geq 0$,

$$p_{0x}^{(2n)} \leq p_{00}^{(2n)}.$$

Noting that $p_{0x}^{(2n)} = 0$ if $|x| > 2n$, we have

$$1 = \sum_{x\in\mathbb{Z}} p_{0x}^{(2n)} \leq (4n+1) \cdot p_{00}^{(2n)},$$

which implies $p_{00}^{(2n)} \geq \frac{1}{4n+1}$ and thereby,

$$\mathbf{E}_0[V_0] = \sum_{n\geq 0} p_{00}^{(n)} = \infty.$$

The Dichotomy Theorem now implies that 0 is recurrent.

(b) Again, we only prove transience of the state 0. Moreover, it suffices to consider $\alpha > 1/2$ by symmetry (since $-X$ is a biased random walk on $\mathbb{Z}$ with parameter $1-\alpha < 1/2$ and 0 is transient for X if and only if it is transient for $-X$).
First, we note that the Markov chain $X = (X_n)_{n\geq 0}$ under $\mathbf{P}_0$ has the same law as $X' = (X'_n)_{n\geq 0}$, defined as the partial sums $X'_0 = 0$ and $X'_n = \sum_{i=1}^n \xi_i$ of a collection $(\xi_i)_{i\geq 1}$ of i.i.d. random variables taking the values $+1$ (resp. -1) with probability α (resp. $1-\alpha$). Therefore, by the strong law of large numbers,

$$\lim_{n\to\infty} \frac{X_n}{n} = \mathbf{E}_0[X_1] = 2\alpha - 1 > 0 \quad \mathbf{P}_0\text{-a.s.}.$$

Hence, for any $\epsilon \in (0, 2\alpha - 1)$,

$$\lim_{N\to\infty} \mathbf{P}_0[\bigcap_{n\geq N} \{X_n \geq \epsilon n\}] = \mathbf{P}_0[\bigcup_{N\geq 0} \bigcap_{n\geq N} \{X_n \geq \epsilon n\}] = 1,$$

and in particular, there exists $N \geq \epsilon^{-1}$ such that

$$1/2 \leq \mathbf{P}_0[\bigcap_{n\geq N} \{X_n \geq \epsilon n\}] \leq \mathbf{P}_0[\bigcap_{n\geq N} \{X_n \geq 1\}] \leq \mathbf{P}_0[\sum_{n\geq 1} \mathbf{1}_{X_n=0} < N] \leq \mathbf{P}_0[V_0 < \infty].$$

By the Dichotomy Theorem, 0 is transient.

For parts (c)-(e), we construct an explicit coupling of reflected random walks and biased random walks for all values of $\alpha \in [0,1]$. On a general probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let $(U_n)_{n\geq 1}$ be a sequence of i.i.d. uniform random variables taking values in $[0,1]$. For every $\alpha \in [0,1]$, we define the following stochastic processes:

(i) $(X_0^{(\alpha)})_{n\geq 0}$ is defined by $X_0^{(\alpha)} = 0$ and for $n \geq 1$ by

$$X_n^{(\alpha)} = X_{n-1}^{(\alpha)} + \mathbb{1}_{U_n \leq \alpha} - \mathbb{1}_{U_n > \alpha}.$$

We note that $(X_0^{(\alpha)})_{n\geq 0}$ is a biased random walk on $\mathbb{Z}$ started at 0, i.e. it is a MC(δ^0,p) with the transition probability given on the exercise sheet.

(ii) $(Y_0^{(\alpha)})_{n\geq 0}$ is defined by $Y_0^{(\alpha)} = 0$ and for $n \geq 1$ by

$$Y_n^{(\alpha)} = \begin{cases} Y_{n-1}^{(\alpha)} + \mathbb{1}_{U_n \leq \alpha} - \mathbb{1}_{U_n > \alpha} & \text{if } Y_{n-1}^{(\alpha)} > 0, \\ Y_{n-1}^{(\alpha)} + 1 & \text{if } Y_{n-1}^{(\alpha)} = 0. \end{cases}$$

We note that $(Y_0^{(\alpha)})_{n\geq 0}$ is a reflected random walk on $\mathbb{N}$ started at 0, i.e. it is a MC(δ^0,p) with the transition probability given on the exercise sheet.

We define

$$H_0^{(\alpha,X)} := \inf\{n \geq 1 : X_n^{(\alpha)} = 0\} \quad \text{and} \quad H_0^{(\alpha,Y)} := \inf\{n \geq 1 : Y_n^{(\alpha)} = 0\},$$

and use analogous notation for $-X$ and $|X|$.

(c) Our goal is to show that $\mathbb{P}[H_0^{(\alpha,Y)} < \infty] = 1$ for $\alpha \leq 1/2$. We will use a comparison to a SRW ($\alpha = 1/2$). More precisely, we first note that for $\alpha \leq 1/2$,

$$H_0^{(\alpha,Y)} \leq H_0^{(1/2,Y)} \quad \mathbb{P}\text{-a.s.},$$

which follows from the definition of $Y^{(\alpha)}$. Moreover, $Y^{(1/2)}$ and $|X^{(1/2)}|$ have the same law. Thus, we conclude that

$$\mathbb{P}[H_0^{(\alpha,Y)} < \infty] \geq \mathbb{P}[H_0^{(1/2,Y)} < \infty] = \mathbb{P}[H_0^{(1/2,|X|)} < \infty] = \mathbb{P}[H_0^{(1/2,X)} < \infty] = 1,$$

where we used Exercise 3.2 (a) in the last equality.

(d) Our goal is to show that $\mathbb{E}[H_0^{(\alpha,Y)}] < \infty$ for $\alpha < 1/2$. We note that conditional on $\{X^{(\alpha)} = 1\}$, the processes $Y^{(\alpha)}$ and $X^{(\alpha)}$ take the same steps up to hitting 0 for the first time. Hence,

$$\begin{aligned} \mathbb{E}[H_0^{(\alpha,Y)}] =& \mathbb{E}[H_0^{(\alpha,X)} | X_1^{(\alpha)} = 1] = \sum_{k\geq 1} \mathbb{P}[H_0^{(\alpha,X)} \geq k | X_1^{(\alpha)} = 1] \\ &\leq \sum_{k\geq 1} \mathbb{P}[X_k^{(\alpha)} \geq 1 | X_1^{(\alpha)} = 1] = \sum_{k\geq 1} \mathbb{P}[\sum_{i=2}^{k} \xi_i \geq 0], \end{aligned}$$

where we set $\xi_i := \mathbb{1}_{U_i \leq \alpha} - \mathbb{1}_{U_i > \alpha}$ to be the steps of $X^{(\alpha)}$. By the law of large numbers, we already know that $\frac{1}{k-1} \sum_{i=2}^{k} \xi_i$ will be close to $\mathbb{E}[\xi_i] = 2\alpha - 1 < 0$ as $k \to \infty$. Quantitative upper bounds on $\mathbb{P}[\sum_{i=2}^{k} \xi_i \geq 0]$ follow from standard large deviation theory (see, e.g. Cramér's theorem). The proof is actually short, and so we include it here. For $\lambda \geq 0$, we obtain

$$\begin{aligned} \mathbb{P}[\sum_{i=2}^{k} \xi_i \geq 0] &= \mathbb{P}[\exp(\lambda \sum_{i=2}^{k} \xi_i) \geq 1] \leq \mathbb{E}[\exp(\lambda \sum_{i=2}^{k} \xi_i)] = \prod_{i=2}^{k} \mathbb{E}[\exp(\lambda \xi_i)] \\ &= \left(e^{\lambda} \cdot \alpha + e^{-\lambda} \cdot (1-\alpha)\right)^{(k-1)}, \end{aligned}$$

where we used Markov's inequality and the independence of the $(\xi_i)_{i\geq 2}$. Minimizing $e^{\lambda} \cdot \alpha + e^{-\lambda} \cdot (1-\alpha)$ over $\lambda \in [0, \infty)$, we obtain that the minimum is attained at $\lambda = \ln(\sqrt{\frac{1-\alpha}{\alpha}}) > 0$ (since $\alpha < 1/2$). Hence,

$$\mathbb{P}[\sum_{i=2}^{k} \xi_i \geq 0] \leq \left(2\sqrt{\alpha(1-\alpha)}\right)^{(k-1)},$$

and so

$$\mathbb{E}[H_0^{(\alpha,Y)}] \leq \sum_{k\geq 1} \left(2\sqrt{\alpha(1-\alpha)}\right)^{(k-1)} = \frac{1}{1 - 2\sqrt{\alpha(1-\alpha)}} < \infty \quad \text{since } \alpha < 1/2.$$

(e) We first note that

$$\mathbb{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1] = \mathbb{P}[H_0^{(1-\alpha,X)} < \infty | X_1 = -1] \geq \mathbb{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1],$$

where we used $\alpha \geq 1/2$ in the inequality. Second, by Exercise 3.2 (b), we know that 0 is transient for the biased random walk with $\alpha > 1/2$, i.e. we obtain

$$\begin{aligned} 1 > \mathbb{P}[H_0^{(\alpha,X)} < \infty] &= \alpha \cdot \mathbb{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1] + (1-\alpha) \cdot \underbrace{\mathbb{P}[H_0^{(\alpha,X)} < \infty | X_1 = -1]}_{\geq \mathbb{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1]} \\ &\geq \mathbb{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1] \end{aligned}$$

By the definition of the processes $Y^{(\alpha)}$ and $X^{(\alpha)}$, we conclude that

$$\mathbb{P}[H_0^{(\alpha,Y)} < \infty] = \mathbb{P}[H_0^{(\alpha,Y)} < \infty | Y_1 = 1] = \mathbb{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1] < 1.$$

Hence, 0 is transient for the reflected random walk with $\alpha > 1/2$.

Solution 3.3

Let $x \in V$. We observe that under $\mathbf{P}_x$, the process $Y = (Y_n)_{n\geq 1}$, defined by

$$Y_n := d(X_n, x),$$

is a reflected random walk on $\mathbb{N}$ starting at 0 with parameter

$$\alpha := \frac{d-1}{d} \geq 2/3,$$

i.e. a Markov chain MC(δ^0, P') with transition probability given by $p'_{0,1} = 1$, $p'_{y,y+1} = \frac{d-1}{d}$, and $p'_{y,y-1} = \frac{1}{d}$ for $y \geq 1$. It now follows from Exercise 3.2 (e) that the state 0 is transient for Y. Clearly, the Markov chain X is in state x if and only if the Markov chain Y is in state 0. Therefore, the state x is transient for X.

Solution 3.4

(a) We have

$$\mathbf{P}_0\left[\left(\max_{0\leq m\leq n} X_m\right) \geq a\right] = \mathbf{P}_0\left[H_a \leq n\right] = \mathbf{P}_0\left[X_n > a\right] + \mathbf{P}_0\left[H_a \leq n, X_n < a\right],$$

where we use that $a \geq 1$ is odd and $n \geq 0$ is even to ensure that $X_n \neq a$ $\mathbf{P}_0$-a.s.

(b) *Idea:* The law of $(X_{H_a+k})_{k\geq 0}$ is the same as the law of $(a - X_{H_a+k})_{k\geq 0}$, i.e. after hitting a at time H_a, we can reflect the trajectory of the SRW since steps $+1$ and -1 both occur with probability $1/2$.

More precisely, we have

$$\mathbf{P}_0\left[H_a \leq n, X_n < a\right] = \sum_{m=0}^{n} \mathbf{P}_0[X_n < a, H_a = m]. \tag{1}$$

By the strong Markov property,

$$\begin{aligned} \mathbf{P}_0[X_n < a, H_a = m] &= \mathbf{P}_0[X_n < a, H_a = m, H_a < \infty] \\ &= \underbrace{\mathbf{P}_0[X_{H_a+n-m} < a, H_a = m | H_a < \infty, X_{H_a} = a]}_{=\mathbf{P}_a[X_{n-m}<a]\cdot \mathbf{P}_0[H_a = m | H_a < \infty, X_{H_a} = a]} \cdot \mathbf{P}_0[H_a < \infty] \\ &= \mathbf{P}_a[X_{n-m} < a] \cdot \mathbf{P}_0[H_a = m]. \end{aligned}$$

Since $\mathbf{P}_a[X_{n-m} > a] = \mathbf{P}_a[X_{n-m} < a]$ by symmetry, we deduce that

$$\begin{aligned}\mathbf{P}_0[X_n < a, H_a = m] &= \mathbf{P}_a[X_{n-m} < a] \cdot \mathbf{P}_0[H_a = m] \\ &= \mathbf{P}_a[X_{n-m} > a] \cdot \mathbf{P}_0[H_a = m] = \mathbf{P}_0[X_n > a, H_a = m],\end{aligned}$$

where we again used the strong Markov property in the last equality. Combined with (1), we conclude that

$$\begin{aligned}\mathbf{P}_0\left[H_a \leq n, X_n < a\right] &= \sum_{m=0}^{n} \mathbf{P}_0[X_n < a, H_a = m] \\ &= \sum_{m=0}^{n} \mathbf{P}_0[X_n > a, H_a = m] = \mathbf{P}_0[X_n > a, H_a \leq n] = \mathbf{P}_0[X_n > a].\end{aligned}$$

Finally, we deduce that

$$\begin{aligned}\mathbf{P}_0\left[\left(\max_{0\leq m\leq n} X_m\right) \geq a\right] &= \mathbf{P}_0\left[X_n > a\right] + \mathbf{P}_0\left[H_a \leq n, X_n < a\right] \\ &= 2\mathbf{P}_0\left[X_n > a\right] = \mathbf{P}_0\left[X_n > a\right] + \mathbf{P}_0\left[X_n < -a\right] \\ &= \mathbf{P}_0[|X_n| > a].\end{aligned}$$

Solution 3.5

(a) Let us denote by $(X_n)_{n\geq 0}$ the Markov chain with transition probability corresponding to the rules of the game. Recall that $H_i = \inf\{n \geq 0; X_n = i\}$. Let us call $k_i = \mathbf{E}_i[H_9]$ for $i \in \{1, \ldots, 9\}$. We observe that 9 is an absorbing state and that $k_9 = 0$. Then we can express H_9 as

$$H_9 = f((X_n)_{n\geq 0}) = \sum_{n=0}^{\infty} 1_{\{X_n<9\}}$$

where f is a measurable function. Then, for $i \in \{1, \ldots, 8\}$ we have $\mathbf{P}_i$-a.s. that

$$\begin{aligned}k_i &= \sum_{j=1}^{9} \mathbf{E}_i[H_9|X_1 = j]\mathbf{P}_i[X_1 = j] \\ &= \sum_{j=1}^{9} \mathbf{E}_i[1_{\{X_0<9\}} + f((X_{n+1})_{n\geq 0})|X_1 = j]p_{i,j} \\ &\overset{(1)}{=} \sum_{j=1}^{9} \left(1 + \mathbf{E}_j[f((X_n)_{n\geq 0})]\right) p_{i,j} \\ &= \sum_{j=1}^{9}(1 + k_j)p_{i,j}\end{aligned}$$

where the equality (1) is justified by the Markov property. Applying this to the model, and considering the effect of the ladders and snakes we get to the following system of equations

$$\begin{aligned}k_1 &= \frac{1}{2}(1 + k_7) + \frac{1}{2}(1 + k_5) \\ k_4 &= \frac{1}{2}(1 + k_5) + \frac{1}{2}(1 + k_1) \\ k_5 &= \frac{1}{2}(1 + k_1) + \frac{1}{2}(1 + k_7) \\ k_7 &= \frac{1}{2}(1 + k_4) + \frac{1}{2}(1 + k_9)\end{aligned}$$

Since $k_9 = 0$ we can solve this system. We obtain that the average number of turns it takes to complete the game is given by $k_1 = 7$.

(b) Notice that the probability that a player starting from the middle square will complete the game without slipping to the square 1 is exactly $\mathbf{P}_5[H_9 < H_1]$. Using the Markov property repeatedly we get

$$\begin{aligned}
\mathbf{P}_5[H_9 < H_1] &= p_{5,6} \underbrace{\mathbf{P}_1[H_9 < H_1]}_{=0} + p_{5,7}\mathbf{P}_7[H_9 < H_1] \\
&= \frac{1}{2}(p_{7,8}\mathbf{P}_4[H_9 < H_1] + p_{7,9} \underbrace{\mathbf{P}_9[H_9 < H_1]}_{=1}) \\
&= \frac{1}{2}\left(\frac{1}{2}(p_{4,5}\mathbf{P}_5[H_9 < H_1] + p_{4,6} \underbrace{\mathbf{P}_1[H_9 < H_1]}_{=0}) + \frac{1}{2}\right) \\
&= \frac{1}{8}\mathbf{P}_5[H_9 < H_1] + \frac{1}{4}
\end{aligned}$$

Then $\mathbf{P}_5[H_9 < H_1] = 2/7$.