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Applied Stochastic Processes

Solution sheet 4

Solution 4.1

(a) The state a is recurrent since $\mathbf{P}_a[H_a < \infty] \geq \mathbf{P}_a[H_a = 1] = 1$. The state b is transient since $\mathbf{P}_b[H_b < \infty] = \mathbf{P}_b[X_1 = b] = 1/3 < 1$.

(b) There are three communitation classes: $C_1 = \{a\}$, $C_2 = \{b, c\}$ and $C_3 = \{d, e\}$. The classes C_1 and C_2 are transient, the class C_3 is recurrent.

(c) No. By definition, an irreducible Markov chain has exactly one communication class.

(d) Yes. For example, take S to be countably infinite and define the transition probability P by $p_{xx} = 1$ for all $x \in S$. Then every state is recurrent and forms its own communication class.

Solution 4.2

(a) For all $\varepsilon \geq 0$, a and b communicate since $p_{ab} = p_{ba} = \frac{1}{2}$.

(b) For all $\varepsilon > 0$, b and d communicate since $p_{bd}^{(2)} = \varepsilon \cdot \frac{2}{3}$ and $p_{db}^{(2)} = \frac{1}{3} \cdot \varepsilon$. For $\varepsilon = 0$, it holds that $p_{bd}^{(n)} = 0$ for all $n \geq 0$, so b and d do not communicate.

(c) The Markov chain is irreducible if and only if $\varepsilon > 0$. This can be seen as in part (b).

Solution 4.3

(a) First, we note that $X_0 \sim \delta^1$. Second, we note that for all $n \geq 0$ and for all $k > 0$ and $\ell \in \mathbb{N}$, it follows from the definition of X_{n+1} that

$$\begin{aligned}\mathbb{P}[X_{n+1} = \ell | X_n = k] &= \mathbb{P}\left[\sum_{i=1}^{k} Z_i^{n+1} = \ell\right] \\ &= \sum_{z_1,\dots,z_k \geq 0:\ z_1+\dots+z_k=\ell} \mathbb{P}\left[Z_1 = z_1, \dots, Z_k = z_k\right] \\ &= \sum_{z_1,\dots,z_k \geq 0:\ z_1+\dots+z_k=\ell} \nu(z_1) \cdot \ldots \cdot \nu(z_k) \\ &=: p_{kl}.\end{aligned}$$

For $k = 0$, we have $\mathbb{P}[X_{n+1} = 0 | X_n = 0] = 1 =: p_{00}$ for all $n \geq 1$. Moreover, by the independence of the $(Z_i^n)_{i,n \geq 1}$'s, we have for all $x_0, \dots, x_{n+1}$,

$$\mathbb{P}[X_{n+1} = x_{n+1} | X_0 = x_0, \dots, X_n = x_n] = \mathbb{P}[X_{n+1} = x_{n+1} | X_n = x_n],$$

whenever $\mathbb{P}[X_0 = x_0, \dots, X_n = x_n] > 0$. Consequently,

$$\mathbb{P}[X_0 = x_0, \dots, X_{n+1} = x_{n+1}] = \delta_{x_0}^1 \cdot p_{x_0 x_1} \cdot \ldots \cdot p_{x_n x_{n+1}},$$

and so we have shown that X is a Markov chain $\mathrm{MC}(\delta^1, P)$ with the transition probability $P = (p_{xy})_{x,y \in \mathbb{N}}$ defined above.

(b) $C_0 = \{0\}$ is a closed communication class and $C_1 = \{1, 2, \dots\}$ is a communication class that is not closed. Since $C_0 \cup C_1 = E$, there are no other communication classes.

To see that $C_0 = \{0\}$ is a closed communication class, it suffices to note that $p_{00} = 1$. To see that $C_1 = \{1, 2, \dots\}$ is a communication class that is not closed, we make the following observations: First, using $\nu(0), \nu(1) > 0$, we have $p_{i,i-1} > 0$ for all $i \geq 1$, and so $i \to i-1$. Second, using $\nu(0) + \nu(1) < 1$, there exists $k \geq 2$ such that $\nu(k) > 0$, and so for all $i \geq 1$, $p_{i,ki} > 0$ and $i \to ki$. Combining these observations, $i \to j$ for all $i \geq 1$ and $j \geq 0$.

(c) C_0 is recurrent since $\mathbf{P}_0[H_0 = 1] = 1$. C_1 is transient, which follows from Corollary 2.14 in Section 2.10. More precisely, since for $i \geq 1$, $i \to 0$ but $0 \not\to i$, it follows that i is transient.

(d) If $\nu(0) = 0$ and $\nu(1) < 1$, then $X_{n+1} \geq X_n$ almost surely for all $n \geq 0$ and $X_{n+1} > X_n$ with positive probability. Consequently, there are infinitely many communication classes: $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$, etc. As before, the class C_0 is recurrent and closed. The classes C_1, $C_2, \dots$ are transient and not closed.

Solution 4.4

(a) Since X_n can take values in $\{0, \dots, N\}$, we set $S := \{0, \dots, N\}$.

On the one hand, for $x < N$, we set

$$p_{x,x+1} = 1 - \frac{x}{N},$$

as in order for X_n to grow by 1, the randomly selected particle must be from container B; this occurs with probability

$$\frac{\#\text{ of particles in } B}{\#\text{ of total particles}} = \frac{N-x}{N}.$$

On the other hand, for $x > 0$, the only other option is for the amount of particles in A to decrease by 1, which happens with probability $\frac{x}{N}$, and so

$$p_{x,x-1} = \frac{x}{N}.$$

Whenever $|x - y| \neq 1$ for $x, y \in S$, we set $p_{xy} = 0$. It can easily be verified that $P = (p_{xy})_{x,y \in S}$ defines a trasition probability.

(b) Our goal is to identify a stationary distribution; this would represent the equilibrium distribution of particles. To this end, we try to find a reversible distribution π. By Proposition 3.1, we know that it would also be stationary.

By definition of reversibility, π needs to satisfy for all $x \in \{0, \dots, N-1\}$,

$$\pi_x p_{x,x+1} = \pi_{x+1} p_{x+1,x}.$$

We use this to calculate π_x explicitly and see if this defines a proper distribution.

$$\pi_{x+1} = \frac{\pi_x (1 - \frac{x}{N})}{\frac{x+1}{N}} = \pi_x \frac{N-x}{x+1} \overset{\text{(Induction)}}{=} \pi_0 \frac{N \cdots (N-x)}{(x+1)!}. \tag{1}$$

Thus we find that $\pi_x = \binom{N}{x}\pi_0$. Since π should define a distribution, we must have $\sum_{x \in S} \pi_x = 1$. Hence we find

$$\pi_0 = \left(\sum_{x \in S} \binom{N}{x} \right)^{-1} = \frac{1}{2^N}. \tag{2}$$

Hence,

$$\pi_x = \binom{N}{x}\frac{1}{2^N},$$

the binomial distribution; which is (as we have shown) reversible, and thus stationary.

Solution 4.5

(a) First, we note that by definition, $\mu_x(y) \geq 0$ for every $y \in S$. Second, we prove the stationarity of μ_x, i.e. for every $y \in S$, $\mu_x(y) = \sum_{z\in E}\mu_x(z)p_{zy}$. Using the hint and the simple Markov property, we obtain

$$\begin{aligned}\sum_{z\in S}\mu_x(z)p_{zy} &= \sum_{z\in S}\sum_{n=0}^{\infty}\mathbf{P}_x[X_n = z, H_x > n]\cdot p_{zy}\\ &= \sum_{z\in S}\sum_{n=0}^{\infty}\mathbf{P}_x[X_n = z, X_{n+1} = y, H_x > n].\end{aligned}$$

By Fubini,

$$\begin{aligned}\sum_{z\in E}\mu_x(z)p_{zy} &= \sum_{n=0}^{\infty}\sum_{z\in S}\mathbf{P}_x[X_n = z, X_{n+1} = y, H_x > n]\\ &= \sum_{n=0}^{\infty}\mathbf{P}_x[X_{n+1} = y, H_x > n]\\ &= \mathbf{E}_x[\sum_{n=0}^{H_x-1}\mathbf{1}_{X_{n+1}=y}] = \mu_x(y),\end{aligned}$$

where we have used in the last equality that $H_x < \infty$ and $X_0 = X_{H_x} = x$ $\mathbf{P}_x$-a.s.

Remark: If x would be transient, then $\mathbf{P}_x[H_x < \infty] < 1$ and so the last equality fails if $x = y$.

Finally, we show that for every $y \in S$, $\mu_x(y) < \infty$. If $x \not\to y$, then $\mu_x(y) = 0$. Otherwise, we have $x \to y$, which implies $y \to x$ since x is recurrent. Hence, $p_{yx}^{(n)} > 0$ for some $n = n(y) \geq 1$. By the stationarity of μ_x,

$$1 = \mu_x(x) = \sum_{y\in S}\mu_x(y)p_{yx}^{(n)},$$

and so $\mu_x(y) < \infty$.

(b) If $x \not\to y$, then $\mu_x(y) = 0$. If $x \to y$, then we obtain

$$\begin{aligned}\mu_x(y) &= \mathbf{E}_x[\sum_{n=0}^{H_x-1}\mathbf{1}_{X_n=y}] = \sum_{k=1}^{\infty}\mathbf{P}_x[\sum_{n=0}^{H_x-1}\mathbf{1}_{X_n=y} \geq k] = \sum_{k=1}^{\infty}\mathbf{P}_x[H_y < H_x]\cdot\mathbf{P}_y[H_y < H_x]^{k-1}\\ &= \mathbf{P}_x[H_y < H_x]\cdot\sum_{k=0}^{\infty}\mathbf{P}_y[H_y < H_x]^k = \frac{\mathbf{P}_x[H_y < H_x]}{\mathbf{P}_y[H_x < H_y]}.\end{aligned}$$