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Applied Stochastic Processes

Solution sheet 5

Solution 5.1

(a) For all $\varepsilon \geq 0$, $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is not reversible. It suffices to note that

$$\pi_c p_{cd} = \frac{1}{6} \neq \frac{1}{12} = \pi_d p_{dc}.$$

(b) $\pi = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2})$ is reversible if and only if $\varepsilon = 0$. For $\varepsilon > 0$, we note that

$$\pi_b p_{bc} = \frac{1}{8} \cdot \varepsilon \neq \frac{1}{4} \cdot \varepsilon = \pi_c p_{cb}.$$

For $\varepsilon = 0$, we note that $\pi_a p_{ab} = \pi_b p_{ba} = \frac{1}{16}$ and $\pi_c p_{cd} = \pi_d p_{dc} = \frac{1}{6}$.

(c) $\pi = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2})$ is stationary if and only if $\varepsilon = 0$. For $\varepsilon = 0$, it follows directly from (e) since reversible implies stationary. For $\varepsilon > 0$,

$$\pi_c = \frac{1}{4} \neq \frac{1}{4} - \frac{\varepsilon}{8} = \pi_b p_{bc} + \pi_c p_{cc} + \pi_d p_{dc}.$$

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(d) $\pi = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$ is stationary for all $\varepsilon \geq 0$. To this end, it suffices to check that π is a left eigenvector associated to the eigenvalue 1 for the matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} - \varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & \frac{1}{3} - \varepsilon & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Solution 5.2

(a) Yes, it is possible. For example, consider the Markov chain with state space $\{a, b, c, d\}$ and transition probability given by

$$p_{ab} = 1,\ p_{bc} = 1,\ p_{cd} = 1,\ p_{dd} = \frac{2}{3},\ \text{and}\ p_{da} = \frac{1}{3}.$$

It follows that

$$\mathbf{E}_a[H_b] = 1,\ \mathbf{E}_a[H_c] = 2,\ \mathbf{E}_a[H_d] = 3,\ \text{and}\ \mathbf{E}_a[H_a] = 3 + \left(\frac{1}{3}\right)^{-1},$$

and so

$$\sum_{x \in \{a,b,c,d\}} \frac{1}{\mathbf{E}_a[H_x]} = 2.$$

(b) No, it is not possible. If P is transient or null recurrent, then for all $y \in S$, $\mathbf{E}_y[H_y] = +\infty$, and so the sum is equal to 0. If P is positive recurrent, then by the theorem in Section 3.3, there exists a unique stationary distribution, given by

$$\pi(y) = \frac{1}{\mathbf{E}_y[H_y]}$$

for all $y \in S$, and so the sum is equal to 1.

(c) Since S is finite, there exists a recurrent state by the proposition in Section 2.13. By irreducibility, P is recurrent, and so it is positive recurrent by the proposition in Section 2.12. By the theorem in Section 3.3, there exists a unique stationary distribution.

(d) First, we note that the transition probability $P = (p_{xy})_{x,y\in S}$ is given by

$$p_{xy} = \mathbf{P}_x[X_1 = y] = \mathbb{P}[X_1 = y|X_0 = x] = \mathbb{P}[X_1 = y] > 0.$$

Hence, the Markov chain is irreducible. To see that it is positive recurrent, we compute

$$\begin{aligned}\mathbf{E}_x[H_x] = \mathbb{E}[H_x|X_0 = x] &= \sum_{k=1}^{\infty} k\cdot\mathbb{P}[X_0 = x, X_1 \neq x, \dots X_{k-1}\neq x, X_k = x|X_0 = x]\\ &= \sum_{k=1}^{\infty} k\cdot\mathbb{P}[X_0\neq x]^{k-1}\cdot\mathbb{P}[X_0 = x] = \frac{1}{\mathbb{P}[X_0 = x]} < \infty\end{aligned}$$

Solution 5.3

First, if π is reversible, then it follows directly from Proposition 3.1 that it is stationary.

Second, we assume that π is stationary and aim to prove that it is reversible. In the case $p = q = 1$, every distribution π is trivially reversible and stationary. In the case $p \neq 1$ or $q \neq 1$, we deduce from $\pi P = \pi$ that

$$\pi_1(1-p) = \pi_2(1-q).$$

Plugging in $\pi_2 = 1-\pi_1$, we get

$$1-q = \pi_1(2-p-q) \quad\iff\quad \pi_1 = \frac{1-q}{2-p-q}.$$

Hence,

$$(\pi_1,\pi_2) = \left(\frac{1-q}{2-p-q}, \frac{1-p}{2-p-q}\right),$$

which is reversible since $\pi_1 p_{12} = \pi_2 p_{21}$.

Solution 5.4

(a) For simplicity of notation, we set $p_{0,1} := p_{N,1}$ and $p_{N+1,N} = p_{1,N}$. For every $i \in \{1,\dots,N\}$,

$$\pi_i = \frac{1}{N} = \alpha\frac{1}{N} + (1-\alpha)\frac{1}{N} = \pi_{i-1}p_{i-1,i} + \pi_{i+1}p_{i+1,i},$$

and so the distribution π is stationary.

Clearly, the biased random walk on $\{1,\dots,N\}$ is irreducible. By the theorem in Section 3.3, the stationary distribution is unique if it exists (i.e. if P is positive recurrent). Hence, π is the unique stationary distribution.

(b) Since any reversible distribution is stationary, it suffices to check if π (which is the the unique stationary distribution as shown in part (a)) is reversible. For $i \in \{1,\dots,N\}$, we have

$$\pi_i p_{i,i+1} = \frac{1}{N}\alpha = \frac{1}{N}(1-\alpha) = \pi_{i+1}p_{i+1,i}$$

if and only if $\alpha = \frac{1}{2}$. Hence, π is reversible if and only if $\alpha = \frac{1}{2}$.

Solution 5.5

(a) Note that for every $x \in S$, we have

$$\sum_{y\in S} \widetilde{p}_{xy} = \delta + (1-\delta)\cdot \sum_{y\in S} p_{xy} = 1.$$

Since $\widetilde{p}_{xy} \geq 0$ for every $x, y \in S$, we conclude that $\widetilde{P}$ is a transition probability.

(b) Let $x, y \in S$ be different states. Since P is irreducible, we know that there exists $x = x_0, x_1, \ldots, x_n = y$ with $p_{x_i x_{i+1}} > 0$ and such that all the x_i's are different. This implies that

$$\widetilde{p}^{(n)}_{x,y} \geq \widetilde{p}_{x_0x_1} \cdots \widetilde{p}_{x_{n-1}x_n} = (1-\delta)^n p_{x_0x_1} \cdots p_{x_{n-1}x_n} > 0,$$

i.e., $\widetilde{P}$ is irreducible. Since $\widetilde{p}_{xx} \geq \delta > 0$, we conclude that $\widetilde{P}$ is aperiodic.

(c) Since P is irreducible and positive recurrent, there exists a unique stationary distribution π for P. We show that π is also a stationary distribution for $\widetilde{P}$, which implies that $\widetilde{P}$ is positive recurrent. Indeed, for every $x \in S$,

$$\sum_{y\in S} \pi(y)\widetilde{p}_{yx} = \delta \cdot \underbrace{\sum_{y\in S} \pi(y)\mathbb{1}_{x=y}}_{=\pi(x)} + (1-\delta)\cdot \underbrace{\sum_{y\in S} \pi(y) p_{yx}}_{=\pi(x)} = \pi(x),$$

where we have used the stationarity of π for P.

Solution 5.6

(a) We check that the two properties in the definition of transition probability are satisfied. For every $x, y \in S$, the definition directly implies $\hat{p}_{xy} \geq 0$. For every $x \in S$ with $\pi(x) = 0$,

$$\sum_{y\in S} \hat{p}_{xy} = \sum_{y\in S} \mathbf{1}_{x=y} = 1,$$

and for every $x \in S$ with $\pi(x) > 0$,

$$\sum_{y\in S} \hat{p}_{xy} = \sum_{y\in S} \frac{\pi(y)p_{yx}}{\pi(x)} = \frac{\pi(x)}{\pi(x)} = 1$$

by stationarity of π.

(b) Let $f, g \in \mathrm{L}^\infty(S)$.

$$\begin{aligned}\langle Pf, g\rangle_\pi &= \sum_{x\in S} (Pf)(x)\, g(x)\, \pi(x) = \sum_{x,y\in S} \pi(x)\, p_{xy}\, f(y)\, g(x) \\ &= \sum_{x,y\in S} \pi(y)\, \hat{p}_{yx}\, f(y)\, g(x) = \sum_{y\in S} f(y)\, (\hat{P}g)(y)\, \pi(y) = \langle f, \hat{P}g\rangle_\pi\end{aligned}$$

In the third equality, we have used that the definition of $\hat{P}$ implies $\pi(x)p_{xy} = \pi(y)\hat{p}_{yx}$ for all $x, y \in S$. If $\pi(y) > 0$, this is clear. If $\pi(y) = 0$, we must have $\pi(x) = 0$ or $p_{xy} = 0$ by stationarity of π.

(c) If π is reversible, then $\pi(y)\hat{p}_{yx} = \pi(x)p_{xy} = \pi(y)p_{yx}$, and so the same calculation as in part (b) shows that $\langle Pf, g\rangle_\pi = \langle f, Pg\rangle_\pi$, i.e. P is self-adjoint.

Fix arbitrary $x, y \in S$ and set $f := \delta_x$ and $g := \delta_y$. If P is self-adjoint, then

$$\pi(y)p_{yx} = \langle Pf, g\rangle_\pi = \langle f, Pg\rangle_\pi = \pi(x)p_{xy},$$

and so p is reversible.