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Applied Stochastic Processes

Solution sheet 6

Solution 6.1

(a) A simple example is provided by taking $S = \{1, 2, 3, 4, 5\}$ and the transition probability P defined by

$$p_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \mod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, P is irreducible and the set

$$\{n \geq 1 : p_{11}^{(n)} > 0\} = \{5, 10, 15, \ldots\}$$

has greatest common divisor equal to 5.

(b)
- Since $p_{aa}^{(3)} > 0$ and $p_{aa}^{(5)} > 0$, it follows that $d_a = 1$.
- Since $a \leftrightarrow b$, $d_a = 1$ implies $d_b = 1$.
- Since $p_{cc}^{(2)} > 0$ and $p_{cc}^{(3)} > 0$, it follows that $d_c = 1$.
- Noting that $p_{dd}^{(n)} > 0$ if and only if n is even, it follows that $d_d = 2$.
- Noting that $p_{ee}^{(n)} > 0$ if and only if n is a multiple of 3, it follows that $d_e = 3$.
- Since $p_{ff}^{(n)} = 0$ for all $n \geq 1$, it follows that $d_f = +\infty$.

(c) Let P denote the transition probability of the biased random walk $X = (X_n)_{n\geq 0}$ on $\mathbb{Z}$. Note that the Markov chain $X = (X_{2n})_{n\geq 0}$, which makes two steps of the biased random walk at every step, has state space $2\mathbb{Z}$ and transition probability P^2.

Claim: P^2 is aperiodic, irreducible. Moreover, P^2 is null recurrent or transient.

Before proving the claim, let us show how to conclude from there. By Theorem 3.15, it then follows that $\lim_{n\to\infty} \mathbf{P}_0[X_{2n} = 0] = 0$. Clearly, $\mathbf{P}_0[X_n = 0] = 0$ for all n odd, and so it follows that $\lim_{n\to\infty} \mathbf{P}_0[X_n = 0] = 0$.

Proof of Claim: Let $i, j \in \mathbb{Z}$ with $i \geq j$. Then $p_{2i,2j}^{2|i-j|} \geq \alpha^{2|i-j|}$ and $p_{2j,2i}^{2|i-j|} \geq (1-\alpha)^{2|i-j|}$, thus P^2 is irreducible on $2\mathbb{Z}$. Moreover, $p_{00}^2 = 2\alpha(1-\alpha) > 0$, and so the P^2 is aperiodic. By Theorem 3.2, it suffices to show that there exists no stationary distribution π for P^2 in order to conclude that P^2 is null recurrent or transient. Note that a stationary distribution π for P^2 needs to satisfy for all $i \in \mathbb{Z}$,

$$\pi_{2i} = \pi_{2(i-1)}\alpha^2 + \pi_{2i}2\alpha(1-\alpha) + \pi_{2(i+1)}(1-\alpha)^2$$
$$\iff \left(\frac{\alpha}{1-\alpha}\right)^2 \cdot \underbrace{(\pi_{2i} - \pi_{2(i-1)})}_{=:\Delta_i} = \underbrace{\pi_{2(i+1)} - \pi_{2i}}_{=:\Delta_{i+1}}.$$

We note that $\Delta_1 = 0$ implies $\Delta_i = 0$ for all $i \in \mathbb{Z}$, and so $(\pi_{2i})_{i\in\mathbb{Z}}$ is constant. Similarly, $\alpha = 1/2$ implies that $(\pi_{2i})_{i\in\mathbb{Z}}$ is constant. But this contradicts the fact that π is a distribution. Thus, we may assume that $\Delta_1 > 0$ and $\alpha \neq 1/2$. Now, $\alpha > 1/2$ implies $\lim_{i\to\infty} \Delta_i = \infty$, and $\alpha < 1/2$ implies $\lim_{i\to-\infty} \Delta_i = \infty$. Again, this contradicts the fact that π is a distribution. In summary, we conclude that there exists no stationary distribution for P^2.

Solution 6.2 We note that the three-state Markov chain X is irreducible and aperiodic. Moreover, Propositions 2.16 and 2.17 imply that the Markov chain X is positive recurrent. Thus, by Theorem 3.2,

$$\lim_{n\to\infty} \mathbf{P}_b[X_n = b] = \pi_b,$$

where π denotes the unique stationary distribution. It remains to find π statisfying $\pi = \pi P$. We compute a left eigenvector associated to the eigenvalue 1 of the transition probability P and obtain $(2/3, 2/3, 1)$. Normalizing yields

$$\pi = (2/7, 2/7, 3/7),$$

and so the limit is $\pi_b = 2/7$.

Solution 6.3 The Markov chain $(X_{2n})_{n\geq 0}$ with $X_0 = c$ has state space $S' = \{a, c\}$ and transition probability P' given by

$$p'_{aa} = \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} = \frac{4}{9}, \quad p'_{ac} = \frac{5}{9}, \quad p'_{cc} = \frac{4}{9}, \text{ and } \quad p'_{ca} = \frac{5}{9}.$$

Clearly, P' is irreducible and aperiodic. By symmetry, the unique stationary distribution is given by $\pi = (1/2, 1/2)$. Thus, by Theorem 3.15,

$$\lim_{n\to\infty} \mathbf{P}_c[X_{2n} = a] = \frac{1}{2}.$$

Solution 6.4

(a) Recall that the space of all configurations is $\{0,1\}^V$. We will use the notation

$$\xi = (\xi(a), \xi(b), \xi(c), \xi(d))$$

throughout this exercise. There are seven admissible configurations given by

$$S = \{(0,0,0,0), (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,0,1,0), (0,1,0,1)\}.$$

(b) Recall that at each step of the Markov chain, a vertex in $v \in V$ is chosen uniformly at random, and if both neighbors are **not** occupied, then $\xi(v)$ is sampled using a fair coin. This yields the following transition probabilities:

$$\begin{aligned}
&p_{(0,0,0,0),(0,0,0,0)} = 1/2,\\
&p_{(0,0,0,0),(1,0,0,0)} = p_{(0,0,0,0),(0,1,0,0)} = p_{(0,0,0,0),(0,0,1,0)} = p_{(0,0,0,0),(0,0,0,1)} = 1/8,\\
&p_{(1,0,0,0),(1,0,0,0)} = 1/2 + 1/2\cdot 1/2 = 3/4,\\
&p_{(1,0,0,0),(0,0,0,0)} = p_{(1,0,0,0),(1,0,1,0)} = 1/8,\\
&p_{(0,1,0,0),(0,1,0,0)} = 1/2 + 1/2\cdot 1/2 = 3/4,\\
&p_{(0,1,0,0),(0,0,0,0)} = p_{(0,1,0,0),(0,1,0,1)} = 1/8,\\
&p_{(0,0,1,0),(0,0,1,0)} = 1/2 + 1/2\cdot 1/2 = 3/4,\\
&p_{(0,0,1,0),(0,0,0,0)} = p_{(0,0,1,0),(1,0,1,0)} = 1/8,\\
&p_{(0,0,0,1),(0,0,0,1)} = 1/2 + 1/2\cdot 1/2 = 3/4,\\
&p_{(0,0,0,1),(0,0,0,0)} = p_{(0,0,0,1),(0,1,0,1)} = 1/8,\\
&p_{(1,0,1,0),(1,0,1,0)} = 1/2 + 1/2\cdot 1/2 = 3/4,\\
&p_{(1,0,1,0),(1,0,0,0)} = p_{(1,0,1,0),(0,0,1,0)} = 1/8,\\
&p_{(0,1,0,1),(0,1,0,1)} = 1/2 + 1/2\cdot 1/2 = 3/4,\\
&p_{(0,1,0,1),(0,1,0,0)} = p_{(0,1,0,1),(0,0,0,1)} = 1/8.
\end{aligned}$$

Representing this transition probability as a directed graph yields that every state has an arrow pointing to itself and there are arrows between two states whenever the two states differ in one coordinate.

(c) It follows directly from (b) that there is a unique communication class, hence P is irreducible. Moreover, since every state has an arrow pointing to itself, P is aperiodic.

Solution 6.5

(a) We note that for every admissible state $\xi \in S$,

$$p_{\xi,\xi} \geq 1/2.$$

Indeed, if a vertex $v \in V$ is picked that has an occupied neighbor, then the state remains unchanged, and if a vertex $v \in V$ is picked that has no occupied neighbor, then the state remains unchanged with probability 1/2.

This implies directly that ξ has period 1 for every $\xi \in S$.

(b) Let us denote by $0 \in S$ the configuration with no particles, and let $\xi \in S$ be any admissible configuration. Define the set of occupied coordinates as $A_\xi := \{i \in V : \xi(i) = 1\}$. We note that

$$p_{0,\xi}^{|A_\xi|} = p_{\xi,0}^{|A_\xi|} = \left(\frac{|A_\xi|}{64} \cdot \frac{1}{2}\right) \cdot \left(\frac{|A_\xi| - 1}{64} \cdot \frac{1}{2}\right) \cdot \ldots \cdot \left(\frac{1}{64} \cdot \frac{1}{2}\right) > 0.$$

Indeed, in order to transition from 0 to ξ (resp. from ξ to 0) in exactly $|A_\xi|$ steps, we have to pick in each step a vertex which is currenctly not occupied but which is occupied in ξ and then sample the new value to be 1.

Hence, $0 \longleftrightarrow \xi$ for every $\xi \in S$, and so P is irreducible.

(c) There are different ways to simulate Z, a uniform random variable in S_k. We will describe three alternatives:

1. We could use the hardcore model as described in Section 3.9. Starting from any fixed admissible configuration, we could first let the Markov chain run up to some large time, and then stop it at the next time it reaches a state in S_k. This yields a random variable Z that is (close to) uniform on S_k. Indeed, we have for every $\xi \in S_k$,

$$\lim_{n\to\infty} \mathbb{P}[X_n = \xi | X_n \in S_k] = \lim_{n\to\infty} \frac{\mathbb{P}[X_n = \xi]}{\mathbb{P}[X_n \in S_k]} = \frac{1/|S|}{|S_k|/|S|} = \frac{1}{|S_k|}.$$

2. Inspired by the hardcore model, we could define a new Markov chain on S_k with stationary distribution π, the uniform distribution on S_k, as follows: We start from a fixed admissible configuration $X_0 = \eta \in S_k$. For $n \geq 0$, we define X_{n+1} from X_n as follows:
   - Pick one of the k particles uniformly at random, i.e. a vertex $v \in V$ with $\xi(v) = 1$. In addition, Pick a vertex $u \in V$ with no particle, i.e. $\xi(u) = 0$, uniformly at random.
   - If u has an occupied neighbor $w \neq v$ in X_n, we do nothing and set $X_{n+1} = X_n$.
   - If none of the neighbors $w \neq v$ of u is occupied in X_n, then we set $X_{n+1}(u) = 1$, $X_{n+1}(v) = 0$, and $X_{n+1}(w) = X_n(w)$ for all $w \notin \{u, v\}$, i.e. we move the particle from u to v.

   If k is not too large, then the Markov chain is irreducible and aperiodic. Note that if $k = 32$, i.e. half of the vertices are occupied, then the only two admissible configurations are the two chessboard configurations, and in this case, the Markov chain will always remain in the initial state. This explains, why we need k to be not too large.

As in the proof of Proposition 3.17, one can verify that the transition probability is symmetric, hence the uniform distribution on S_k is reversible, thus stationary. This implies the desired result.

3. Inspired by the hardcore model, we could define a new Markov chain on S_k with stationary distribution π, the uniform distribution on S_k, as follows: We start from a fixed admissible configuration $X_0 = \eta \in S_k$. For $n \geq 0$, we define X_{n+1} from X_n as follows:

   - Pick a pair of vertices $(u, v) \in V \times V$ uniformly at random.
   - If the configuration with $X_{n+1}(v) = X_n(u)$, $X_{n+1}(u) = X_n(v)$, and $X_{n+1}(w) = X_n(w)$ for all $w \notin \{u, v\}$ is admissible, we make this change, i.e. we interchange u and v. Otherwise, we set $X_{n+1} = X_n$, i.e. do nothing.

   If k is not too large, then the Markov chain is irreducible and aperiodic. Again, note that if $k = 32$, i.e. half of the vertices are occupied, then the only two admissible configurations are the two chessboard configurations, and in this case, the Markov chain will always remain in the initial state. This explains, why we need k to be not too large.

   As in the proof of Proposition 3.17, one can verify that the transition probability is symmetric, hence the uniform distribution on S_k is reversible, thus stationary. This implies the desired result.