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Applied Stochastic Processes

Solution sheet 7

Solution 7.1

(a) Yes, by the law of large numbers for renewal processes,

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu} \quad \text{a.s.}$$

(b) Yes, by the law of large numbers for renewal processes,

$$\lim_{t\to\infty} \frac{N_t}{t^2} = \lim_{t\to\infty} \underbrace{\frac{N_t}{t}}_{\to\frac{1}{\mu}\in[0,\infty)} \cdot \underbrace{\frac{1}{t}}_{\to 0} = 0 \quad \text{a.s.}$$

(c) Yes, by the law of large numbers for renewal processes,

$$\lim_{t\to\infty} \frac{N_t}{\sqrt{t}} = \lim_{t\to\infty} \underbrace{\frac{N_t}{t}}_{\to\frac{1}{\mu}\in(0,\infty)} \cdot \underbrace{\sqrt{t}}_{\to+\infty} = +\infty \quad \text{a.s.}$$

(d) Since $\mu = \infty$, $\lim_{t\to\infty} \frac{N_t}{t} = 0$ a.s., and thus we cannot apply the law of large numbers to determine the limit of $N_t/\sqrt{t}$. In fact, the behaviour of $N_t/\sqrt{t}$ for large t depends on the arrival distribution F.

Solution 7.2

(a) Since $T_1 = 1$ a.s., we also obtain $S_k = T_1 + \ldots + T_k = k$ a.s., and so for every $t \geq 0$,

$$N_t = \sum_{k=1}^{\infty} \mathbf{1}_{S_k \leq t} = \sum_{k=1}^{\infty} \mathbf{1}_{k \leq t} = \lfloor t \rfloor.$$

In particular, $\mathbb{E}[N_t] = \lfloor t \rfloor$ for every $t \geq 0$. This function starts at 0, is piecewise constant and makes jumps of height 1 at every integer value of t.

(b) As in the proof of Proposition 5.3, we have for every $t \geq 0$,

$$m(t) = \mathbb{E}[N_t] = \sum_{k=1}^{\infty} \mathbb{P}[T_1 + \ldots + T_k \leq t] = \sum_{k=1}^{\infty} F^{*k}(t).$$

We now focus on computing F^{*k} for $k \geq 1$. Since $T_1 \sim \mathcal{U}(0,1)$, its cumulative distribution function F is given by $F(t) = t \cdot \mathbf{1}_{0\leq t\leq 1} + \mathbf{1}_{t>1}$. We note that F has density $f(t) = \mathbf{1}_{0\leq t\leq 1}$. This allows us to compute for $k = 2$,

$$\begin{aligned}(F * F)(t) &= \int_0^t F(t-s)dF(s) = \int_0^t F(t-s)\mathbf{1}_{0\leq s\leq 1}ds = \int_0^{\min\{1,t\}} (t-s)\cdot \mathbf{1}_{0\leq t-s\leq 1} + \mathbf{1}_{t-s>1}ds \\ &= \int_0^{\min\{1,t\}} (t-s)\cdot \mathbf{1}_{t-1\leq s\leq t} + \mathbf{1}_{s<t-1}ds.\end{aligned}$$

For $t \in [0,1]$, we obtain

$$(F * F)(t) = \int_0^t (t-s)ds = [st - s^2/2]_0^t = t^2/2,$$

and in the same way, by iteration, also

$$F^{*k}(t) = t^k/k!.$$

Summing up, we conclude that for $t \in [0,1]$,

$$m(t) = \sum_{k=1}^{\infty} t^k/k! = e^t - 1.$$

This allows us to draw the function for $t \in [0,1]$. Computations for larger values of t are possible but require more care. We also note that the renewal equation provides an alternative way to compute $m(t)$.

(c) Using Proposition 4.1, we have for every $t \geq 0$, $N_t \sim \text{Poisson}(2t)$, and so $m(t) = 2t$. This function starts at 0 and is linear with slope 2.

(d) Using Proposition 4.2 with $\alpha = 1$ and $\beta = 1/2$, we have for every $t \geq 0$,

$$N_t \sim X_0 + \sum_{i=1}^{\lfloor t \rfloor} (1 + X_i),$$

where the X_i's are i.i.d. geometric variables with parameter 1/2. Since X_i has expectation $\frac{1-\beta}{\beta} = 1$, we have $m(t) = 1 + 2\lfloor t \rfloor$. This function starts at 1, is piecewise constant and makes jumps of height 2 at every integer value of t.

Solution 7.3

Let Φ denote the distribution function of the standard normal distribution, and let $\lceil x \rceil$ be the smallest integer greater than or equal to x for $x \in \mathbb{R}$. Let $S_n := \sum_{i=1}^n T_i$, then using the central limit theorem we have

$$\lim_{n\to\infty} \mathbb{P}[(S_n - n\mu)/\sigma\sqrt{n} < x] = \Phi(x)$$

uniformly in $x \in \mathbb{R}$. Note that Φ is continuous and so it does not matter whether we consider the event with strict inequality $<$ or weak inequality $\leq$ on the left side.

For simplicity of notation, we define

$$Z_t := \frac{N_t - t/\mu}{\sigma\sqrt{t/\mu^3}}.$$

Now, for given $t > 0$ and $x \in \mathbb{R}$, since N_t is integer-valued, we have

$$\mathbb{P}[Z_t < x] = \mathbb{P}\left[N_t < \lceil x(\sigma\sqrt{t/\mu^3}) + t/\mu \rceil\right].$$

Setting $h(t) := \lceil x(\sigma\sqrt{t/\mu^3}) + t/\mu \rceil$, from

$$\{N_t < h(t)\} = \{S_{h(t)} > t\}$$

we obtain that

$$\mathbb{P}[Z_t < x] = \mathbb{P}[S_{h(t)} > t] = \mathbb{P}\left[(S_{h(t)} - \mu h(t))/\sigma\sqrt{h(t)} > (t - \mu h(t))/\sigma\sqrt{h(t)}\right].$$

It suffices to show $h(t) \to \infty$ and $z(t) := (t - \mu h(t))/\sigma\sqrt{h(t)} \to -x$ as $t \to \infty$, since in that case the *uniform convergence* in the central limit theorem will imply

$$\mathbb{P}\left[(S_{h(t)} - \mu h(t))/\sigma\sqrt{h(t)} > z(t)\right] \to 1 - \Phi(-x) = \Phi(x),$$

which means that $\mathbb{P}[Z_t < x] \to \Phi(x)$ and therefore Z_t converges to the standard normal distribution in law as $t \to \infty$. Indeed, if a sequence of functions $(f_n)_{n\geq 1}$ converges *uniformly* to a *continuous* function f, and a sequence of real numbers $(y_n)_{n\geq 1}$ converges to some $y \in \mathbb{R}$, then one can easily prove that $\lim_{n\to\infty} f_n(y_n) = f(y)$. Now for any sequence $(t_n)_{n\geq 1}$ tending to infinity, we can define f_n as the distribution function of $(S_{h(t_n)} - \mu h(t_n))/\sigma\sqrt{h(t_n)}$ and $y_n := z(t_n)$. Since f_n converges uniformly to the function $f(x) := 1 - \Phi(x)$ and y_n converges to $y := -x$, using the above claim we can deduce the desired result.
The fact that $\lim_{t\to\infty} h(t) = \infty$ is easy to see. To show that $\lim_{t\to\infty} z(t) = -x$, we first note that by definition $h(t) = x(\sigma\sqrt{t/\mu^3}) + t/\mu + \epsilon(t)$, where $|\epsilon(t)| < 1$, and hence

$$\begin{aligned} z(t) &= \frac{t - \mu[x(\sigma\sqrt{t/\mu^3}) + t/\mu + \epsilon(t)]}{\sigma\sqrt{h(t)}} \\ &\sim \frac{-\mu x(\sigma\sqrt{t/\mu^3})}{\sigma\sqrt{t/\mu}} \\ &\to -x \text{ as } t \to \infty. \end{aligned}$$

Solution 7.4

(a) First, we note that the set $A := \{a' > 0 : \mathbb{P}[T_1 \in a'\mathbb{Z}] = 1\}$ is non-empty and bounded since T_1 is lattice and takes values in $\mathbb{R}$. Second,

$$b := \min\{b' > 0 : \mathbb{P}[T_1 = b'] > 0\}$$

is well-defined. Indeed, the set $B := \{b' > 0 : \mathbb{P}[T_1 = b'] > 0\}$ is non-empty since $\mathbb{P}[T_1 > 0] > 0$ and T_1 is lattice. Furthermore, $\inf B$ is attained as a minimum because for any $a' \in A$,

$$\inf B = \inf\{b' \in a'\mathbb{Z}_{>0} : \mathbb{P}[T_1 = b'] > 0\},$$

and $a'\mathbb{Z}_{>0} \subset \mathbb{R}$ is a closed set that is bounded from below. We also note that b is a multiple of a' for any $a' \in A$. Finally, we set

$$k^* := \min\{k \geq 1 : b/k < \sup A\}.$$

If $\sup A$ is *not* attained, then we can choose $\tilde{a} \in A$ satisfying $b/k^* < \tilde{a} < \sup A$. But this contradicts our previous observation that $\tilde{a}$ divides b. Hence, $\sup A$ is attained and a is well-defined.

(b) Since $(N_t)_{t\geq 0}$ is a renewal process with jump times in $a\mathbb{Z}$, it directly follows that $\widetilde{N}_t := N_{at}$ defines a renewal process with integer-valued jump times.

(c) We first note that for all $i \in S$, $\mathbb{P}[T_1 = i] \geq 0$. Thus, $p = (p_{ij})_{i,j\in S}$ is well-defined and by definition, $p_{ij} \geq 0$ for all $i, j \in S$. Furthermore, for $i \geq 1$,

$$\sum_{j\in S} p_{ij} = p_{i,i-1} = 1,$$

and for $i = 0$,

$$\sum_{j\in S} p_{0j} = \sum_{j\geq 1} \mathbb{P}[T_1 = j] = 1,$$

since $\mathbb{P}[T_1 = 0] = 0$. Hence, P is a transition probability.

Case 1: $S = \{0, 1, \dots, N-1\}$

The chain is irreducible since $p_{0,N-1} = \mathbb{P}[T_1 = N] > 0$ and for every $j \in \{0, 1, \dots, N-1\}$, we have $p_{N-1,j}^{(N-1-j)} = 1$. Furthermore, the hitting time satisfies $H_0 \le N$ and so the chain is recurrent.

Case 2: $S = \mathbb{N}$

We first note that $\mathbf{P}_0[H_0 = +\infty] = \mathbb{P}[T_1 = +\infty] = 0$, and so the state 0 is recurrent. Furthermore, for every $i \ge 1$, there exists some (minimal) $j \ge i$ such that $\mathbb{P}[T_1 = j] > 0$, and so we have

$$p_{0i}^{(j-i)} = p_{0,j-1} \cdot \prod_{k=1}^{j-i-1} p_{j-k,j-k-1} = \mathbb{P}[T_1 = j] > 0.$$

Hence, $0 \to i$, and in fact, $0 \leftrightarrow i$ by the recurrence of 0. This concludes that the chain is irreducible and recurrent.

Before we show that the chain is aperiodic, we note that for any $k \in \mathbb{N}$ (satisfying $k \le N$ if $n < \infty$),

$$\mathbf{P}_0[H_0 = j] = p_{0,j-1} \cdot \left(\prod_{k=1}^{j-1} p_{j-k,j-k-1} \right) = \mathbb{P}[T_1 = j].$$

Hence, the law of H_0 under $\mathbf{P}_0$ is the same as the law of T_1 under $\mathbb{P}$. Finally, let d be the period of the state 0 (and therefore of the chain P). By definition, we have that $p_{00}^{(n)} = 0$ for all $n \notin d\mathbb{Z}$. Hence, $H_0 \in d\mathbb{Z}$ $\mathbf{P}_0$-a.s. and equivalently, $T_1 \in d\mathbb{Z}$ $\mathbb{P}$-a.s.. This implies that $d = 1$ since $d \ge 2$ would contradict $a = 1$.

(d) For any $t \ge 0$,

$$m(t) = \mathbb{E}[N_t] = \mathbb{E}\left[\sum_{i \ge 1} \mathbf{1}_{T_1 + \dots + T_i \le t} \right] = \mathbf{E}_0 \left[\sum_{n=1}^{\lfloor t \rfloor} \mathbf{1}_{X_n = 0} \right] = \sum_{n=1}^{\lfloor t \rfloor} p_{00}^{(n)}$$

By the theorem on the density of visit times for Markov chains (Sections 3.7 – 3.8),

$$\lim_{t \to \infty} \frac{1}{\lfloor t \rfloor} \sum_{n=1}^{\lfloor t \rfloor} p_{00}^{(n)} = \frac{1}{\mathbf{E}_0[H_0]}.$$

Hence,

$$\lim_{t \to \infty} \frac{m(t)}{t} = \lim_{t \to \infty} \frac{m(t)}{\lfloor t \rfloor} = \frac{1}{\mathbb{E}[T_1]} = \frac{1}{\mu}.$$

(e) For $s \le t$, the computation from part (d) shows that

$$m(t) - m(s) = \sum_{n=\lfloor s \rfloor + 1}^{\lfloor t \rfloor} p_{00}^{(n)}$$

By the results on the convergence of aperiodic, irreducible Markov chains (Section 2.8), we have

$$p_{00}^{(n)} = \mathbf{P}_0[X_n = 0] \longrightarrow \frac{1}{\mathbf{E}_0[H_0]} = \frac{1}{\mu} \quad \text{as } n \to \infty.$$

Since for $k \in \mathbb{N}$ the interval $(t, t+k]$ contains exactly k integers, we conclude that

$$\lim_{t \to \infty} m(t+k) - m(t) = \frac{k}{\mu}.$$