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Applied Stochastic Processes

Solution sheet 8

Solution 8.1

(a) Since $T_1 \sim \text{Exp}(\lambda)$, we have for all $t \geq 0$, $N_t \sim \text{Pois}(\lambda t)$ by Proposition 4.1. Hence, $m(t) = \mathbb{E}[N_t] = \lambda t = t/\mu$. So yes, this is true.

(b) No, it is not true. It suffices to show that for $t \leq 1$,

$$m(t) = \mathbb{E}[N_t] = \sum_{k\geq 1} \mathbb{P}[N_t \geq k] = \sum_{k \geq 1} \underbrace{\mathbb{P}[T_1 + \ldots + T_k \leq t]}_{=t^k/k!} = e^t - 1 \neq 2t = t/\mu.$$

Here, we have used that $\mathbb{P}[T_1 + \ldots + T_k \leq t] = t^k/k!$ for $t \leq 1$, which can easily be proven by induction. Indeed, $\mathbb{P}[T_1 \leq t] = t$ and then

$$\begin{aligned}\mathbb{P}[T_1 + \ldots + T_k \leq t] &= \int_0^t \mathbb{P}[T_1 + \ldots + T_{k-1} \leq t - s] ds \\ &= \int_0^t (t-s)^{k-1}/(k-1)!\, ds = t^k/k!.\end{aligned}$$

(c) Yes, this follows from the elementary renewal theorem.

(d) If the arrival distribution is non-lattice, then this follows from Blackwell's renewal theorem. If the arrival distribution is lattice, the statement is not true in general (see Exercise 7.4 (e) for a correct statement in this case).

Solution 8.2

For clarity of notation, we denote by $\widetilde{G}_i$ the Lebesgue-Stieltjes measure defined by the function $G_i : \mathbb{R}^+ \to \mathbb{R}^+$ (extended to a right-continuous, non-decreasing function on $\mathbb{R}$ by setting $G_i(t) = 0$ for $t < 0$). In particular, $\widetilde{G}_i(\{0\}) = G_i(0)$. For $0 \leq s \leq t$, we write $\int_s^t h(r) d\widetilde{G}_i(r)$ for the integral over $[s,t]$ and $\int_{s^+}^t h(r) d\widetilde{G}_i(r)$ for the integral over $(s,t]$.

(a) For $t \geq 0$,

$$\begin{aligned}\lim_{t' \searrow t} G_1 * G_2(t') &= \lim_{t' \searrow t} \int_0^t G_1(t'-s) d\widetilde{G}_2(s) + \underbrace{\lim_{t'\searrow t} \int_{t^+}^{t'} G_1(t'-s) d\widetilde{G}_2(s)}_{=0} \\ &= \int_0^t \underbrace{\lim_{t'\searrow t} G_1(t'-s)}_{=G_1(t-s)} d\widetilde{G}_2(s) = G_1 * G_2(t),\end{aligned}$$

where we have used the monotonicity and right-continuity of G_1. This establishes the right-continuity of $G_1 * G_2$. To obtain monotonicity, we note that for $t' \geq t \geq 0$,

$$G_1 * G_2(t') - G_1 * G_2(t) = \int_{t^+}^{t'} \underbrace{G_1(t'-s)}_{\geq 0} d\widetilde{G}_2(s) + \int_0^t \underbrace{(G_1(t'-s) - G_1(t-s))}_{\geq 0} d\widetilde{G}_2(s) \geq 0.$$

Finally, for $t \geq 0$,

$$\begin{aligned} G_1 * G_2(t) &= \int_0^t G_1(t-s)d\widetilde{G}_2(s) = \int_0^t \int_0^{t-s} d\widetilde{G}_1(r)d\widetilde{G}_2(s) \\ &= \int_{\{(r,s)\in[0,t]^2\,:\,r+s\leq t\}} d(\widetilde{G}_1 \otimes \widetilde{G}_2)(r,s) \\ &= \int_0^t \int_0^{t-r} d\widetilde{G}_2(s)d\widetilde{G}_1(r) = G_2 * G_1(t), \end{aligned}$$

where we have used Tonelli's theorem and we have written $\widetilde{G}_1 \otimes \widetilde{G}_2$ for the product measure.

(b) First, by part (a), $G_1 * G_2$ is a right-continuous, non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, and so the associated Lebesgue-Stieltjes measure $\widetilde{G_1 * G_2}$ is well-defined. Second, we define the measurable map $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) := x + y$ and claim that $\widetilde{G_1 * G_2} = f\#(\widetilde{G}_1 \otimes \widetilde{G}_2)$, where $f\#(\widetilde{G}_1 \otimes \widetilde{G}_2)$ denotes the push-forward of the product measure of $\widetilde{G}_1$ and $\widetilde{G}_2$. Indeed, for $t \geq 0$ we have as in part (a)

$$\widetilde{G_1 * G_2}([0,t]) = G_1 * G_2(t) = \int_{\{(r,s)\,:\,0\leq r+s\leq t\}} d(\widetilde{G}_1 \otimes \widetilde{G}_2)(r,s),$$

and so the measure agree on sets of the form $[0,t]$. This extends to sets of the form $(s,t]$ by taking differences and to $\mathcal{B}(\mathbb{R})$ by a Dynkin argument. Finally, for $h : \mathbb{R}^+ \to \mathbb{R}$ and $t \geq 0$,

$$\begin{aligned} (h * G_1) * G_2(t) &= \int_0^t (h * G_1)(t-s)dG_2(s) = \int_0^t \int_0^{t-s} h(t-s-r)d\widetilde{G}_1(r)d\widetilde{G}_2(s) \\ &= \int_{\{(r,s)\,:\,0\leq r+s\leq t\}} h(t-(r+s))d(\widetilde{G}_1 \otimes \widetilde{G}_2)(r,s) \\ &= \int_0^t h(t-u)d(\widetilde{G_1 * G_2})(u) = h * (G_1 * G_2)(t), \end{aligned}$$

where we have used the change-of-variables formula and the equality of the measures $\widetilde{G_1 * G_2}$ and $f\#(\widetilde{G}_1 \otimes \widetilde{G}_2)$.

Solution 8.3 Fix $x, t \geq 0$. We can separate $a_x(t)$ into two parts, one for the probability if there has already been a renewal before time t, and one if that hasn't occurred:

$$a_x(t) = \mathbb{P}\left[T_1 > t, A_t \leq x\right] + \mathbb{P}\left[T_1 \leq t, A_t \leq x\right]. \tag{1}$$

Now we analyze each term separately. The first term can be directly expressed as

$$\mathbb{P}\left[T_1 > t, t \leq x\right] = \mathbf{1}_{t\leq x}(1 - F(t)). \tag{2}$$

For the second term, we exploit the renewal structure of the process. Observe that A_t is measurable with respect to $(T_1, T_2, \ldots)$: by definition, we have $A_t = \psi_t(T_1, T_2, \ldots)$, where

$$\psi_t(t_1, t_2, \ldots) = \sum_{n\geq 0} \mathbf{1}_{t_1+\cdots+t_n\leq t,\, t_1+\cdots+t_{n+1}>t}(t - (t_1 + \cdots + t_n)). \tag{3}$$

Notice that for every $s \leq t$, $\psi_t(s, t_2, \ldots) = \psi_{t-s}(t_2, \ldots)$. Using this observation, we find

$$\begin{aligned}
\mathbb{P}\left[T_1 \leq t, A_t \leq x\right] =& \mathbb{P}\left[T_1 \leq t, \psi_t(T_1, T_2, \ldots) \leq x\right] && (4)\\
=& \int_0^t \mathbb{P}\left[\psi_t(s, T_2, \ldots) \leq x\right] dF(s) && (5)\\
=& \int_0^t \mathbb{P}\left[\psi_{t-s}(T_2, \ldots) \leq x\right] dF(s) && (6)\\
=& \int_0^t a_x(t-s) dF(s) = (a_x * F)(t) && (7)
\end{aligned}$$

Thus $a_x(t) = \mathbf{1}_{t \leq x}(1 - F(t)) + (a_x * F)(t)$.

Solution 8.4 Let $S_k = T_1 + \ldots + T_k$. For $t \geq 0$,

$$\begin{aligned}
g(t) &= \mathbb{P}[Y_t = 1] \\
&= \mathbb{P}[Y_t = 1, T_1 > t] + \mathbb{P}[Y_t = 1, T_1 \leq t] \\
&= \mathbb{P}[U_1 > t] + \mathbb{E}\left[\textstyle\sum_{k \geq 0} 1_{\{S_k \leq t < S_k + U_{k+1}\}} 1_{\{T_1 \leq t\}}\right] \\
&= \mathbb{P}[U_1 > t] + \mathbb{E}\left[\textstyle\sum_{k \geq 1} 1_{\{T_1 + S_k - S_1 \leq t < T_1 + S_k - S_1 + U_{k+1}\}}\right],
\end{aligned}$$

where T_1 is independent of $S_k - S_1$ and of U_{k+1}, and $S_k - S_1 \overset{(d)}{=} S_{k-1}$ for $k \geq 1$. This implies that

$$\begin{aligned}
g(t) &= \mathbb{P}[U_1 > t] + \int_0^t \mathbb{E}\left[\textstyle\sum_{k \geq 1} 1_{\{S_{k-1} \leq t - s < S_{k-1} + U_{k+1}\}}\right] dF(s) \\
&= h(t) + \int_0^t g(t-s) dF(s).
\end{aligned}$$