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Applied Stochastic Processes

Solution sheet 9

Solution 9.1

(a) Let us define $h(t) = \mathbb{1}_{\{t \le x\}}(1 - F(t))$ for $t \ge 0$. Note that $h \ge 0$, it is measurable, continuous a.e. and bounded by 1. Also it vanishes outside the compact interval $[0, x]$. This implies that h is directly Riemann integrable. Since F is non-lattice, by Smith's key renewal theorem it follows that

$$\lim_{t\to\infty} a_x(t) = \frac{1}{\mathbb{E}[T_1]} \int_0^\infty h(t)dt = \frac{1}{\mu} \int_0^x (1 - F(t))dt =: G(x).$$

(b) To see that G is a distribution function, we note that

$$\lim_{x\to\infty} G(x) = \frac{1}{\mu} \int_0^\infty \mathbb{P}[T_1 > t]dt = \frac{\mathbb{E}[T_1]}{\mu} = 1.$$

This means that A_t converges in distribution to a random variable with distribution G.

Solution 9.2

(a) Note that $h \ge 0$ and it is a non increasing function. Also

$$\int_0^\infty h(t)dt = \int_0^\infty \mathbb{P}[U_1 > t]dt = \mathbb{E}[U_1] < \infty,$$

which means that h is directly Riemann integrable. Since F is non-lattice and g is solution of the equation $g = h + g * F$, we know by Smith's key renewal theorem that

$$\lim_{t\to\infty} g(t) = \frac{1}{\mathbb{E}[T_1]}\mathbb{E}[U_1] = \frac{\mathbb{E}[U_1]}{\mathbb{E}[U_1] + \mathbb{E}[V_1]}.$$

Solution 9.3

For fixed $k \ge 0$,

$$\mathbb{P}[X_n = k] = \frac{n!}{k!(n-k)!} p_n^k (1 - p_n)^{n-k} \tag{1}$$

$$= \underbrace{\frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{n \cdot n \cdot \ldots \cdot n}}_{\xrightarrow{n\to\infty} 1} \cdot \frac{1}{k!} \cdot \underbrace{(p_n \cdot n)^k}_{\xrightarrow{n\to\infty} \lambda^k} \big(1 - \frac{p_n \cdot n}{n}\big)^{n-k}, \tag{2}$$

and since $\frac{p_n \cdot n}{n} \cdot (n-k) \to \lambda$, one has $(1 - \frac{p_n \cdot n}{n})^{n-k} \to e^{-\lambda}$ as $n \to \infty$. Hence,

$$\mathbb{P}[X_n = k] \longrightarrow \frac{e^{-\lambda}}{k!}\lambda^k = \mathbb{P}[X = k], \tag{3}$$

and for $y \in \mathbb{R}$,

$$F_{X_n}(y) = \mathbb{P}[X_n \le y] = \sum_{k \le y} \mathbb{P}[X_n = k] \xrightarrow{n\to\infty} \sum_{k \le y} \mathbb{P}[X = k] = \mathbb{P}[X \le y] = F_X(y). \tag{4}$$