Mathematics for New Technologies in Finance

Exercise sheet 2

Through this exercise sheet, we let $E=\mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J\colon \mathcal{C}_0^1(J,E)\to \mathbf{T}(E)$ the signature map such that for all $X\in\mathcal{C}_0^1(J,E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X)=(1,\mathbf{s}_1,\cdots,\mathbf{s}_M)\in\mathbf{T}^{(M)}(E)$. Let $X\in\mathcal{C}_0^1([0,s],E)$ and $Y\in\mathcal{C}_0^1([s,t],E)$.

Exercise 2.1 (Signatures)

(a) Let $X_t=t\mathbf{x}\in\mathbb{R}^d$ for all $t\in[0,1]$. Calculate $\mathbf{Sig}_{[0,1]}(X)$.

(b) Let $X\in\mathcal{C}_0^1([0,T],E)$ and $X_0=0$. Prove that

$$\mathbf{Sig}_{[0,1]}(X)_{1,2}+\mathbf{Sig}_{[0,1]}(X)_{2,1}=\mathbf{Sig}_{[0,1]}(X)_1\cdot\mathbf{Sig}_{[0,1]}(X)_2. \tag{1}$$

Exercise 2.2 (Calculate Signatures)

(a) Let $X\in\mathcal{C}_0^1([0,1],\mathbb{R})$ s.t. $X_t=\sin(t)$ for all $t\in[0,1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(b) Let $X\in\mathcal{C}_0^1([0,1],\mathbb{R}^2)$ s.t. $X_t=(t,\sin(t))$ for all $t\in[0,1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(c) Let $X\in\mathcal{C}_0^1([0,1],\mathbb{R})$ and $n\in\mathbb{N}$. Calculate $\int_0^1 t^n dX_t$ when

  (i) $X_t=t$

  (ii) $X_t=\sin(t)$

(d) Prove that

$$\mathcal{F}=\left\{\mathcal{C}_0^1([0,1],\mathbb{R})\ni X\mapsto\sum_{i=1}^n\lambda_i\int t^i dX_t\in\mathbb{R}\colon \forall\lambda_i\in\mathbb{R},n\in\mathbb{N}\right\}$$

is a point-separating vector space. $\mathcal{C}_0^1([0,1],\mathbb{R})$ is the space of all function f on $[0,1]$ with $f(0)=0$ and f has continuous derivative.

References

[1] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv preprint arXiv:1603.03788*, 2016.

[2] Terry J Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Springer, 2007.