Mathematics for New Technologies in Finance

Exercise sheet 3

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J : \mathcal{C}_0^1(J, E) \to \mathbf{T}(E)$ the signature map for all $X \in \mathcal{C}_0^1(J, E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X) = (1, \mathbf{s}_1, \cdots, \mathbf{s}_M) \in \mathbf{T}^{(M)}(E)$.

Exercise 3.1 (Controlled ODEs) Consider the controlled ODE: $X_0 = x \in \mathbb{R}$

$$dX_t^\theta = V^\theta(t, X_t^\theta)dt, \quad t \in [0, T]. \tag{1}$$

(a) Let

$$a_t = \frac{\partial X_T^\theta}{\partial X_t^\theta}. \tag{2}$$

Prove that

$$\frac{d}{dt} a_t = -\frac{\partial V^\theta}{\partial x}(t, X_t^\theta) \cdot a_t, \quad a_T = 1, \tag{3}$$

and relate a_t with $J_{t,T}$ in the lecture notebook.

(b) Prove that

$$\frac{d}{dt}\left(\frac{\partial X_t^\theta}{\partial \theta} a_t\right) = a_t \frac{\partial V^\theta}{\partial \theta}(t, X_t^\theta), \tag{4}$$

and

$$\frac{\partial X_T^\theta}{\partial \theta} = -\int_T^0 \frac{\partial X_T^\theta}{\partial X_t^\theta} \cdot \frac{\partial V^\theta}{\partial \theta}(t, X_t^\theta)dt. \tag{5}$$

(c) Is every feedforward neural network a discretization of controlled ODE?

Exercise 3.2 (Linear controlled ODE) Let $E = \mathbb{R}^d, W = \mathbb{R}^n$. Let $X \in \mathcal{C}_0^1([0, T], E)$ and let $B : E \to \mathbf{L}(W)$ be a bounded linear map. Consider

$$dY_t = B(dX_t)(Y_t) \tag{6}$$

If we denote $B^k = B(e_k)$, $k = 1, \cdots, d$ then

$$dY_t = \sum_{k=1}^{d} B^k(Y_t)dX_t^k. \tag{7}$$

Prove that

$$Y_t = \Big(\sum_{k=0}^{\infty} B^{\otimes k}\Big)\Big(\mathbf{Sig}_{[0,t]}(X)\Big)Y_0. \tag{8}$$

This implies that the solution of controlled SDE could be written as a linear function on signature stream of driving path. This implies that signature stream is a promising feature for controlled ODE.

References

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