Mathematics for New Technologies in Finance

Exercise sheet 4

Exercise 4.1 (Brownian motion) For all $N \in \mathbb{N}$, we split $[0,1]$ into N many intervals with equal length $\delta = \frac{1}{N}$. We define a discrete stochastic processes W^N at $\{0, \delta, 2\delta, \dots, 1\}$ s.t. $W_0^N = 0$ and

$$W_{n\delta+\delta}^N - W_{n\delta}^N = X_n, \quad \forall n = 1, \dots, N,$$

where (X_n) are i.i.d. random variable s.t. $\mathbb{P}(X_n = \sqrt{\delta}) = \mathbb{P}(X_n = -\sqrt{\delta}) = \frac{1}{2}$. W^N is a symmetric random walk with walking size $\sqrt{\delta}$.

(a) Prove that at $t = n\delta$.

$$\mathbb{E}[W_t^N] = 0, \quad \mathrm{Var}[W_t^N] = t. \tag{1}$$

(b) Prove that at each fixed $t = n\delta$, as $n \to \infty$.

$$W_t^N \to \mathcal{N}(0, t). \tag{2}$$

(c) What is the definition of Brownian motion?

(d) Prove that Brownian motion is $\frac{1}{2} - \epsilon$ Hölder continuous a.s. (Hint: Borel-Cantelli Lemma)

Exercise 4.2 (Ito's formula) Let W be a Brownian motion on $[0, \infty)$ and define

$$Q^n(W) = \sum_{i=1}^{n} (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2. \tag{3}$$

(a) Prove that $Q^n(W)$ converges to 1 in L^2

(b) Prove the following convergence in L^2 sense

$$\lim_{n\to\infty} \sum_{i=1}^{n} W_{\frac{i-1}{n}} (W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) = \frac{W_1^2 - 1}{2} \tag{4}$$

(c) Prove that if f is smooth and bounded

$$f(W_t) = f(0) + \int_0^t f'(W_s) dW_s + \int_0^t \frac{f''(W_s)}{2} ds. \tag{5}$$

Exercise 4.3 (Black-Scholes model) Let $\sigma > 0$, $X_t = X_0 \exp\{\sigma W_t - \frac{\sigma^2 t}{2}\}$.

(a) Prove that X is a solution of

$$dX_t = \sigma X_t dW_t.$$

(b) Let $K > 0$, calculate

$$C_0 = \mathbb{E}[(X_T - K)_+].$$

(c) Let $K > 0$, calculate

$$\frac{\partial}{\partial X_0} \mathbb{E}[(X_T - K)_+].$$