Mathematics for New Technologies in Finance

Exercise sheet 8

Exercise 8.1 (Breeden-Litzenberger formula)

(a) Recall the Black-Schole formula

(b) Is there always a positive implied volatility σ_{imp} related to the option price? If yes, prove it. Otherwise, on which price interval there is always a positive implied volatility σ_{imp} related to the option price?

(c) Prove the Breeden-Litzenberger formula:

$$\partial_K^2 C(T,K)dK = \text{law}(S_T)(dK). \tag{1}$$

(d) Discretize the Breeden-Litzenberger formula and link it with Butterfly spreads.

Exercise 8.2 (Dupire formula) Assume the following local volatility model:

$$dS_t = \sigma(t, S_t)S_t dW_t. \tag{2}$$

(a) If $\sigma(t, S_t) = \sigma S_t^\beta$, for which value of β, the market has leverage effect (the volatility increases when the stock price goes down), which is empirically observed.

(b) Let V_t be the fair price of an European payoff $h(S_T)$. Prove the backward Kolmogorov equation:

$$\partial_t V_t + \frac{1}{2}\sigma(S,t)^2 S^2 \partial_{SS}^2 V_t = 0 \tag{3}$$

(c) Let f_T^S be the probability density function of S_T, prove the forward Kolmogorov equation (Fokker-Planck equation):

$$\partial_T f(S,T) = \frac{1}{2}\partial_S^2\Big(\sigma(S,T)^2 S^2 f(S,T)\Big) \tag{4}$$

(d) Prove by Fokker-Planck equation the Dupire formula:

$$\sigma^2(K,T) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2\partial_K^2 C(T,K)} \tag{5}$$

where $C(T,K)$ is the option price of maturity T and strike K.

References

[1] Pierre Henry-Labordère. Calibration of local stochastic volatility models to market smiles: A monte-carlo approach. *Risk Magazine, September*, 2009.