Mathematics for New Technologies in Finance

Exercise sheet 9

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J \colon \mathcal{C}_0^1(J, E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J, E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X) = (1, \mathbf{s}_1, \cdots, \mathbf{s}_M) \in \mathbf{T}^{(M)}(E)$. Let $X \in \mathcal{C}_0^1([0, s], E)$ and $Y \in \mathcal{C}_0^1([s, t], E)$. The concatenated path $X \star Y \in \mathcal{C}_0^1([0, t], E)$ is defined by

$$(X \star Y)_u = \begin{cases} X_u & u \in [0, s] \\ Y_u + (X_s - Y_s) & u \in [s, t] \end{cases} \tag{1}$$

Exercise 9.1 (Basic properties of signatures)

(a) (Invariant under reparametrizatio) Let $X \in \mathcal{C}_0^1([S_1, T_1], E)$ and $\tau \colon [S_2, T_2] \to [S_1, T_1]$ a non-decreasing surjective reparametrization. Then

$$\mathbf{Sig}_{[S_2,T_2]}(X_{\tau(\cdot)}) = \mathbf{Sig}_{[S_1,T_1]}(X). \tag{2}$$

(b) Prove that neither is $\mathbf{Sig}_{[0,1]}$ surjective nor the range of which a linear subspace of $\mathbf{T}(E)$.

(c) Prove that signature of the augmented paths i.e. $\bar{X}_t = (t, X_t)$ is unique.

(d) Prove that signature satisfies the following equation

$$d\mathbf{Sig}_{[0,t]}^{(M)}(X) = \mathbf{Sig}_{[0,t]}^{(M)}(X) \otimes dX_t \tag{3}$$

Exercise 9.2 (Chen's identity)

(a) Prove the Chen's identity:

$$\mathbf{Sig}_{[r,t]}(X \star Y) = \mathbf{Sig}_{[r,s]}(X) \otimes \mathbf{Sig}_{[s,t]}(Y). \tag{4}$$

(b) Prove the Chen's identity for truncated signature:

$$\mathbf{Sig}_{[r,t]}^{(M)}(X \star Y) = \mathbf{Sig}_{[r,s]}^{(M)}(X) \otimes \mathbf{Sig}_{[s,t]}^{(M)}(Y). \tag{5}$$

(c) Let X be linear on $[n, n+1]$ and let $X_{n+1} - X_n = \mathbf{x}_n$ for $n \in \mathbb{N}$, use Chen's identity to prove that

$$\mathbf{Sig}_{[0,N]}(X) = \bigotimes_{n \le N} (1, \mathbf{x}_n, \frac{\mathbf{x}_n^{\otimes 2}}{2!}, \cdots). \tag{6}$$

References

[1] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv preprint arXiv:1603.03788*, 2016.

[2] Terry J Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Springer, 2007.