Mathematics for New Technologies in Finance

Solution sheet 2

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J : \mathcal{C}_0^1(J, E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J, E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X) = (1, \mathbf{s}_1, \cdots, \mathbf{s}_M) \in \mathbf{T}^{(M)}(E)$. Let $X \in \mathcal{C}_0^1([0, s], E)$ and $Y \in \mathcal{C}_0^1([s, t], E)$.

Exercise 2.1 (Signatures)

(a) Let $X_t = t\mathbf{x} \in \mathbb{R}^d$ for all $t \in [0, 1]$. Calculate $\mathbf{Sig}_{[0,1]}(X)$.

(b) Let $X \in \mathcal{C}_0^1([0, T], E)$ and $X_0 = 0$. Prove that

$$\mathbf{Sig}_{[0,1]}(X)_{1,2} + \mathbf{Sig}_{[0,1]}(X)_{2,1} = \mathbf{Sig}_{[0,1]}(X)_1 \cdot \mathbf{Sig}_{[0,1]}(X)_2. \tag{1}$$

Solution 2.1

(a)

$$\mathbf{Sig}_{[0,1]}(X) = (1, \mathbf{x}, \frac{\mathbf{x}^{\otimes 2}}{2!}, \cdots). \tag{2}$$

(b) By integration by part, we directly show the equality

$$\int_0^1 u_t^{(1)} du_t^{(2)} + \int_0^1 u_t^{(2)} du_t^{(1)} = \int_0^1 d(u^{(1)} \cdot u^{(2)})_t = u_1^{(1)} \cdot u_1^{(2)} \tag{3}$$

Exercise 2.2 (Calculate Signatures)

(a) Let $X \in \mathcal{C}_0^1([0, 1], \mathbb{R})$ s.t. $X_t = \sin(t)$ for all $t \in [0, 1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(b) Let $X \in \mathcal{C}_0^1([0, 1], \mathbb{R}^2)$ s.t. $X_t = (t, \sin(t))$ for all $t \in [0, 1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(c) Let $X \in \mathcal{C}_0^1([0, 1], \mathbb{R})$ and $n \in \mathbb{N}$. Calculate $\int_0^1 t^n dX_t$ when

  (i) $X_t = t$

  (ii) $X_t = \sin(t)$

(d) Prove that

$$\mathcal{F} = \left\{ \mathcal{C}_0^1([0, 1], \mathbb{R}) \ni X \mapsto \sum_{i=1}^n \lambda_i \int t^i dX_t \in \mathbb{R} : \forall \lambda_i \in \mathbb{R}, n \in \mathbb{N} \right\}$$

is a point-separating vector space. $\mathcal{C}_0^1([0, 1], \mathbb{R})$ is the space of all function f on $[0, 1]$ with $f(0) = 0$ and f has continuous derivative.

Solution 2.2

(a)

$$\Big(1, \sin(1), \int_0^1 \sin(t)\cos(t)dt\Big) \tag{4}$$

(b)

$$\Big(1, 1, \sin(1), \frac{1}{2}, \int_0^1 \sin(t)dt, \int_0^1 t\cos(t)dt, \int_0^1 \sin(t)\cos(t)dt\Big) \tag{5}$$

(c)

$$\Big(1, 1, \sin(1), \frac{1}{2}, \int_0^1 \sin(t)dt, \int_0^1 t\cos(t)dt, \int_0^1 \sin(t)\cos(t)dt\Big) \tag{6}$$

(d) (i)

$$\frac{t^{n+1}}{n+1}\Big|_0^1 \tag{7}$$

(ii)

$$\begin{aligned}
\int_0^1 t^n d\sin(t) &= \sin(t)t^n\Big|_0^1 + \int_0^1 nt^{n-1} d\cos(t) \\
&= \sin(t)t^n\Big|_0^1 + \int_0^1 nt^{n-1} d\cos(t) \\
&= \sin(t)t^n\Big|_0^1 + n\cos(t)t^{n-1}\Big|_0^1 - \int_0^1 n(n-1)t^{n-2} d\sin(t) \\
&= \dots
\end{aligned} \tag{8}$$

(e) Vector space holds directly from the definition. So we remain to show point-separating. Let us consider $Z \in \mathcal{C}_0^1([0,1], \mathbb{R})$ s.t.

$$\int \sum_{i=1}^n \lambda_i t^i dZ_t = 0, \quad \forall \lambda_i \in \mathbb{R}, n \in \mathbb{N}.$$

An elementary approach is using universal approximation of polynomials. Since Z' is continuous on $[0,1]$, it can be universally approximated by polynomials, and therefore we have

$$\int_0^1 (Z_t')^2 dt = \lim_{n\to\infty} \int \sum_{i=1}^n \lambda_i t^i dZ_t = 0. \tag{9}$$

This implies that $Z = 0$ because it starts from 0, which completes the proof.

Remark: It worth noticing that this essentially relies on that Z' is continuous. But we can actually make the proof more general by considering function X which are only L-Lipschitz and starting from 0, and then a more general proof can be done by fourier analysis. Since $\sin(m\pi t)$ and $\cos(m\pi t)$ for all $m \in \mathbb{N}$ are uniformly approximated by polynomial on $[0,1]$. We have for all $m \in \mathbb{N}$

$$\int \sin(mt)dZ_t = \int \cos(mt)dZ_t = 0 \tag{10}$$

Then we define a sign measure $\mu(dt) = Z_t' dt$ (Because by Rademacher's Lipschitz function is almost everywhere differentiable and here we even know that $|Z_t'| \leq L$ almost surely), then for all $m \in \mathbb{N}$

$$\int \sin(mt)d\mu = \int \cos(mt)d\mu = 0. \tag{11}$$

Then by fourier analysis we know $\mu = 0$ so Z is constant, which is actually 0 because $Z(0) = 0$. This proof uses the same idea used in the proof of universal approximation theory of neural network by G. Cybenko.

References

[1] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv preprint arXiv:1603.03788*, 2016.

[2] Terry J Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Springer, 2007.