Mathematics for New Technologies in Finance

Solution sheet 4

Exercise 4.1 (Brownian motion) For all $N \in \mathbb{N}$, we split $[0,1]$ into N many intervals with equal length $\delta = \frac{1}{N}$. We define a discrete stochastic processes W^N at $\{0, \delta, 2\delta, \dots, 1\}$ s.t. $W_0^N = 0$ and

$$W^N_{n\delta+\delta} - W^N_{n\delta} = X_n, \quad \forall n = 1, \dots, N,$$

where (X_n) are i.i.d. random variable s.t. $\mathbb{P}(X_n = \sqrt{\delta}) = \mathbb{P}(X_n = -\sqrt{\delta}) = \frac{1}{2}$. W^N is a symmetric random walk with walking size $\sqrt{\delta}$.

(a) Prove that at $t = n\delta$.
$$\mathbb{E}[W_t^N] = 0, \quad \mathrm{Var}[W_t^N] = t. \tag{1}$$

(b) Prove that at each fixed $t = n\delta$, as $n \to \infty$.
$$W_t^N \to \mathcal{N}(0,t). \tag{2}$$

(c) What is the definition of Brownian motion?

(d) Prove that Brownian motion is $\frac{1}{2} - \epsilon$ Hölder continuous a.s. for all $\epsilon \in (0, \frac{1}{2})$. (Hint: Borel-Cantelli Lemma)

Solution 4.1

(a)
$$\mathbb{E}[W_t^N] = \mathbb{E}[\sum_{i=1}^n X_i] = 0. \tag{3}$$

By independence
$$\mathrm{Var}[W_t^N] = \mathrm{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathrm{Var}(X_i) = n\delta = t. \tag{4}$$

(b) By CLT
$$\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{\sqrt{n}} \sim \mathcal{N}(0,t) \tag{5}$$

(c) A equivalent definition to the standard one is the following: Let $H = L^2((0,\infty), \mathcal{B}((0,\infty)), \lambda)$ a Hilbert space where λ is the Lesbesgue meansure. Denote by W the H-isonormal Gaussian process and let $B_t = W(\mathbb{1}_{(0,t]})$. Then (B_t) has the law as the Brownian motion and the a.s. continuous path can be proven by the standard Kolmogorov's continuity theorem.

(d) Recall the Kolmogorov's continuity theorem (an application of Borel-Cantelli). If a real-valued stochastic process (X_t) has the property that for all $K > 0$, there exists positive α, β, C such that for all $0 \le t \le t+h \le K$,
$$\mathbb{E}[|X_{t+h} - X_t|^\alpha] \le Ch^{1+\beta}. \tag{6}$$
Then a.s. (X_t) is γ-Hölder-continuous for all $\gamma < \frac{\beta}{\alpha}$. Since for all $k \in \mathbb{N}$ there exists $C > 0$ s.t.
$$\mathbb{E}[|B_{t+h} - B_t|^{2k}] \le Ch^{1+k}. \tag{7}$$
Therefore Brownian motion is $\frac{1}{2} - \epsilon$ Hölder continuous a.s. for all $\epsilon \in (0, \frac{1}{2})$.

Exercise 4.2 (Ito's formula) Let W be a Brownian motion on $[0,\infty)$ and define

$$Q^n(W) = \sum_{i=1}^{n} (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2. \tag{8}$$

(a) Prove that $Q^n(W)$ converges to 1 in L^2

(b) Prove the following convergence in L^2 sense

$$\lim_{n\to\infty} \sum_{i=1}^{n} W_{\frac{i-1}{n}}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) = \frac{W_1^2 - 1}{2} \tag{9}$$

(c) Prove that if f is smooth and bounded

$$f(W_t) = f(0) + \int_0^t f'(W_s)dW_s + \int_0^t \frac{f''(W_s)}{2} ds. \tag{10}$$

Solution 4.2

(a)

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{i=1}^{n}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2 - 1\Big)^2\Big] &= \mathrm{Var}(\sum_{i=1}^{n}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2) \\
&= \sum_{i=1}^{n} \mathrm{Var}((W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2) \\
&= n(\mathbb{E}(W_{\frac{1}{n}}^4) - \frac{1}{n^2}) \\
&= n(\frac{3}{n^2} - \frac{1}{n^2}) \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

(b) Combining (a) and the fact that

$$2W_{\frac{i-1}{n}}(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) = (W_{\frac{i}{n}} + W_{\frac{i-1}{n}})(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) - (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2.$$

(c) By Ito's formula

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$

Then taking integral of both sides gives us the result.

Exercise 4.3 (Black-Scholes model) Let $\sigma > 0$, $X_t = X_0 \exp\{\sigma W_t - \frac{\sigma^2 t}{2}\}$.

(a) Prove that X is a solution of

$$dX_t = \sigma X_t dW_t.$$

(b) Let $K > 0$, calculate

$$C_0 = \mathbb{E}[(X_T - K)_+].$$

(c) Let $K > 0$, calculate

$$\frac{\partial}{\partial X_0}\mathbb{E}[(X_T - K)_+].$$

Solution 4.3

(a) Check by Ito's formula

(b)

$$C_0 = X_0\Phi(d_1) - K\Phi(d_2) \tag{11}$$

where

$$d_1 = \frac{\log(\frac{X_0}{K}) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

We show details in exercise class.

(c)

$$\frac{\partial}{\partial X_0}\mathbb{E}[(X_T - K)_+] = \Phi(d_1).$$

We show details in exercise class.