Mathematics for New Technologies in Finance

Solution sheet 8

Exercise 8.1 (Breeden-Litzenberger formula)

(a) Recall the Black-Schole formula

(b) Is there always a positive implied volatility σ_{imp} related to the option price? If yes, prove it. Otherwise, on which price interval there is always a positive implied volatility σ_{imp} related to the option price?

(c) Prove the Breeden-Litzenberger formula:

$$\partial_K^2 C(T,K)dK = \text{law}(S_T)(dK). \tag{1}$$

(d) Discretize the Breeden-Litzenberger formula and link it with Butterfly spreads.

Solution 8.1

(a)

$$C(T,K) = N(d_1)S_0 - N(d_2)K \tag{2}$$

where

$$d_1 = \frac{\log(S_0/K) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T} \tag{3}$$

(b) Since

$$\partial_\sigma C(T,K) = N'(d_1)\sqrt{T} > 0 \tag{4}$$

we only need to analyze the boundary:

$$\lim_{\sigma\to 0} C(T,K) = (S_0 - K)_+ \tag{5}$$

and

$$\lim_{\sigma\to\infty} C(T,K) = S_0 \tag{6}$$

(c)

$$\begin{aligned}\partial_K^2 C(T,K) &= \partial_K^2 \int (S-K)_+ f(S,T)dS \\ &= \partial_K \int_K^\infty -f(S,T)dS = f(K,T)\end{aligned} \tag{7}$$

(d) Let $K_1 < K_2 < K_3$ Then

$$C(T,K_1) + C(T,K_3) - 2C(T,K_2) \tag{8}$$

is exactly Butterfly spread.

Exercise 8.2 (Dupire formula) Assume the following local volatility model:

$$dS_t = \sigma(t,S_t)S_t dW_t. \tag{9}$$

(a) If $\sigma(t, S_t) = \sigma S_t^{\beta}$, for which value of β, the market has leverage effect (the volatility increases when the stock price goes down), which is empirically observed.

(b) Let V_t be the fair price of an European payoff $h(S_T)$. Prove the backward Kolmogorov equation:

$$\partial_t V_t + \frac{1}{2}\sigma(S,t)^2 S^2 \partial^2_{SS} V_t = 0 \tag{10}$$

(c) Let f_T^S be the probability density function of S_T, prove the forward Kolmogorov equation (Fokker-Planck equation):

$$\partial_T f(S,T) = \frac{1}{2}\partial^2_S\Big(\sigma(S,T)^2 S^2 f(S,T)\Big) \tag{11}$$

(d) Prove by Fokker-Planck equation the Dupire formula:

$$\sigma^2(K,T) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2\partial^2_K C(T,K)} \tag{12}$$

where $C(T,K)$ is the European call option price of maturity T and strike K.

Solution 8.2

(a) $\beta < 0$

(b) By Ito formula we have

$$dV(t,S_t) = \partial_t V(t,S_t)dt + \partial_S V(t,S_t)dS_t + \frac{1}{2}\partial^2_{SS}V(t,S_t)\sigma(t,S_t)^2 S_t^2 dt \tag{13}$$

Since $V_t(S_t)$ is a martingale, terms in front of dt must be 0 which completes the proof.

(c) Since the local volatility model is Markov, we can directly apply the Fokker-Plank equation to it and obtain the result.

(d)

$$\begin{aligned}
\partial_T C(T,K) &= \partial_T \int (S-K)_+ f(S,T)dS \\
&= \int (S-K)_+ \partial_T f(S,T)dS \\
&= \int (S-K)_+ \frac{1}{2}\partial^2_S\Big(\sigma(S,T)^2 S^2 f(S,T)\Big)dS \\
&= \frac{1}{2}\sigma(K,T)^2 K^2 f(K,T) \\
&= \frac{1}{2}\sigma(K,T)^2 K^2 \partial^2_K C(T,K).
\end{aligned} \tag{14}$$

References

[1] Pierre Henry-Labordère. Calibration of local stochastic volatility models to market smiles: A monte-carlo approach. *Risk Magazine, September*, 2009.