Mathematics for New Technologies in Finance

Solution sheet 9

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J\colon \mathcal{C}_0^1(J,E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J,E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X) = (1, \mathbf{s}_1, \cdots, \mathbf{s}_M) \in \mathbf{T}^{(M)}(E)$. Let $X \in \mathcal{C}_0^1([0,s],E)$ and $Y \in \mathcal{C}_0^1([s,t],E)$. The concatenated path $X \star Y \in \mathcal{C}_0^1([0,t],E)$ is defined by

$$(X \star Y)_u = \begin{cases} X_u & u \in [0,s] \\ Y_u + (X_s - Y_s) & u \in [s,t] \end{cases} \tag{1}$$

Exercise 9.1 (Basic properties of signatures)

(a) (Invariant under reparametrization) Let $X \in \mathcal{C}_0^1([S_1,T_1],E)$ and $\tau\colon [S_2,T_2] \to [S_1,T_1]$ a non-decreasing surjective reparametrization. Then

$$\mathbf{Sig}_{[S_2,T_2]}(X_{\tau(\cdot)}) = \mathbf{Sig}_{[S_1,T_1]}(X). \tag{2}$$

(b) Prove that neither is $\mathbf{Sig}_{[0,1]}$ surjective nor the range of which a linear subspace of $\mathbf{T}(E)$.

(c) Prove that signature of the augmented paths i.e. $\bar{X}_t = (t, X_t)$ is unique.

(d) Prove that signature satisfies the following equation

$$d\mathbf{Sig}_{[0,t]}(X) = \mathbf{Sig}_{[0,t]}(X) \otimes dX_t \tag{3}$$

Solution 9.1

(a) Because line integral is invariant under reparametrization and the definition of

(b) Because

$$\mathbf{Sig}_{[0,1]}(X)_{1,2} + \mathbf{Sig}_{[0,1]}(X)_{2,1} = \mathbf{Sig}_{[0,1]}(X)_1 \cdot \mathbf{Sig}_{[0,1]}(X)_2. \tag{4}$$

(c) Because signature is unique under tree like equivalent, so with one augmented coordinate as strictly increasing, the only tree like equivalent path is the augmented path it self .

(d)

$$\begin{aligned} d\mathbf{Sig}_{[0,t_1]}(X) &= d\Big(\int_0^{t_1} \cdots \int_0^{t_m} dX_{t_m}^{i_m} \dots dX_{t_1}^{i_1}\Big)_{(i_1,\dots,i_m)\in\{1,\dots,d\}^m} \\ &= \Big(\int_0^{t_2} \cdots \int_0^{t_m} dX_{t_m}^{i_m} \dots dX_{t_2}^{i_2}\Big)_{(i_2,\dots,i_m)\in\{1,\dots,d\}^{m-1}} (dX_{t_1}^{i_1})_{i_1\in\{1,\dots,d\}} \\ &= \mathbf{Sig}_{[0,t_1]}(X) \otimes dX_{t_1} \end{aligned} \tag{5}$$

Exercise 9.2 (Chen's identity)

(a) Prove the Chen's identity:

$$\mathbf{Sig}_{[r,t]}(X \star Y) = \mathbf{Sig}_{[r,s]}(X) \otimes \mathbf{Sig}_{[s,t]}(Y). \tag{6}$$

(b) Prove the Chen's identity for truncated signature:

$$\mathbf{Sig}^{(M)}_{[r,t]}(X \star Y) = \mathbf{Sig}^{(M)}_{[r,s]}(X) \otimes \mathbf{Sig}^{(M)}_{[s,t]}(Y). \tag{7}$$

(c) Let X be linear on $[n, n+1]$ and let $X_{n+1} - X_n = \mathbf{x}_n$ for $n \in \mathbb{N}$, use Chen's identity to prove that

$$\mathbf{Sig}_{[0,N]}(X) = \bigotimes_{n \leq N} (1, \mathbf{x}_n, \frac{\mathbf{x}_n^{\otimes 2}}{2!}, \cdots). \tag{8}$$

Solution 9.2

(a) By the uniqueness of the solution of linear ode. we only need to check that $\mathbf{Sig}_{[r,s]}(X) \otimes \mathbf{Sig}_{[s,t]}(Y)$ follows the dynamic. On $[0, s]$ it is obvious and on $u \in [s, t]$

$$d\Big(\mathbf{Sig}_{[r,s]}(X)\otimes\mathbf{Sig}_{[s,u]}(Y)\Big) = \mathbf{Sig}_{[r,s]}(X)\otimes\Big(\mathbf{Sig}_{[s,u]}(Y)dY_u\Big) = \Big(\mathbf{Sig}_{[r,s]}(X)\otimes\mathbf{Sig}_{[s,u]}(Y)\Big)dY_u \tag{9}$$

which completes the proof.

(b) Similarly we prove by

$$d\mathbf{Sig}^{(M)}_{[0,t]}(X) = \mathbf{Sig}^{(M)}_{[0,t]}(X) \otimes dX_t \tag{10}$$

(c) Since on each linear segment we have

$$\mathbf{Sig}_{[\frac{n}{N},\frac{n+1}{N}]}(X) = (1, \mathbf{x}_n, \frac{\mathbf{x}_n^{\otimes 2}}{2!}, \cdots). \tag{11}$$

Then by the Chen's theorem we have

$$\mathbf{Sig}_{[0,N]}(X) = \bigotimes_{n \leq N} (1, \mathbf{x}_n, \frac{\mathbf{x}_n^{\otimes 2}}{2!}, \cdots). \tag{12}$$

References

[1] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv preprint arXiv:1603.03788*, 2016.

[2] Terry J Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*. Springer, 2007.