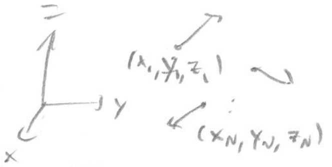
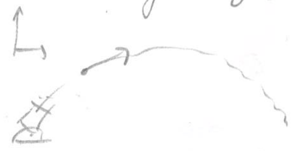
From classical mechanics to Floer's geometry

→ NOT FORGET! Assign to everybody a talk!

1. Hamiltonian mechanics

A classical physical system may be described in terms of generalized coordinates and their associated speeds:





Free particle(s)	... in gravity	... in a magnetic field	Pendulum
$q = (x_i, y_i, z_i)_{i=1}^N$	$q = (x, y)$	$q = (\rho, \theta, z)$	$q = \theta$
$\dot q = (\dot x_i, \dot y_i, \dot z_i)_{i=1}^N$	$\dot q = (\dot x, \dot y)$	$\dot q = (\dot\rho, \dot\theta, \dot z)$	$\dot q = \dot\theta$
$T = \frac{1}{2}\sum_i m_i(\dot x_i^2 + \dot y_i^2 + \dot z_i^2)$ $V = 0$	$T = \frac{1}{2}m(\dot x^2 + \dot y^2)$ $V = mgy$	$T = \frac{1}{2}m(\dot\rho^2 + \dot z^2)$ $V = Q(U(q,t) - \dot q \cdot \vec A(q,t))$	$T = \frac{1}{2}m l^2 \dot\theta^2$ $V = -mgl\cos\theta$

Moral: The state of a system can be thought as a point $x \in TN$, where N is a manifold described by the generalized coordinates (configuration space). We will suppose that $N \subseteq \mathbb{R}^{3N}$ is closed without boundary. (holonomic constraints)

Q: What about motion?

Axiom: (Principle of least action) Let $L := T - V : [0,1] \times TN \longrightarrow \mathbb{R}$ be the Lagrangian of a system. The path $x : [0,1] \longrightarrow TN$ describing the physical evolution from $x(0) = x_0$ to $x(1) = x_1$ is given by an extremum of the action functional

$$S : \mathcal{P}(x_0, x_1) \longrightarrow \mathbb{R}$$
$$x \longmapsto \int_0^1 L(t, x(t), \dot x(t))\,dt.$$

L- A path $x : [0,1] \longrightarrow TN$ is an extremum of S iff

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} = \frac{\partial L}{\partial q_i} \quad i = 1, \dots, n \quad \text{(Euler-Lagrange eqns)}$$

in local coordinates.

Proof (in $\mathbb{R}^{2n}$): We have an identification $T_x\mathcal{P}(x_0, x_1) = P(0,0)$, then

$$(dS)_x(\xi) = \frac{d}{d\varepsilon} S(x + \varepsilon\xi)\Big|_{\varepsilon=0}$$
$$= \int_0^1 \frac{d}{d\varepsilon} L(t, x(t) + \varepsilon\xi(t), \dot x(t) + \varepsilon\dot\xi(t))\Big|_{\varepsilon=0} dt$$
$$= \int_0^1 \sum_i \left[\frac{\partial L}{\partial q_i}\xi_i + \frac{\partial L}{\partial \dot q_i}\dot\xi_i\right] dt$$

$$= (dS)_x(\xi) \overset{\text{int.}}{\underset{\text{part}}{=}} \int_0^1 \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} \right] \xi_i \, dt .$$

Thus, $dS_x = 0 \iff \dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot q_i} = 0 \quad \forall i.$

Ex.: For the projectile, we have $L = \frac{1}{2}m(\dot x^2 + \dot y^2) - mgy$. The E-L equations then become

$$\begin{cases} m\ddot x = 0 \\ m\ddot y = -mg, \end{cases}$$

so that $(x, y) = (\dot x_0 t + x_0, -gt^2 + \dot y_0 t + y_0)$.

Suppose now that L satisfies the Legendre condition:

$$\forall (t, q) \in [0,1] \times N, \quad L_t|_{T_qN} : T_qN \to \mathbb{R} \text{ is s. convex,}$$

i.e., $\det\left(\dfrac{\partial^2 L}{\partial \dot q_i \partial \dot q_j} \right) > 0$ in local coordinates. Note that

$$d(L_t|_{T_qN})_{\dot q} : \underset{\substack{\cong \\ T_qN}}{T_{\dot q} T_q N} \longrightarrow \underset{\substack{\cong \\ \mathbb{R}}}{T_{L(t,q,\dot q)} \mathbb{R}},$$

so that we can take the Legendre transformation as

$$\begin{aligned} \mathcal{L}L_t : TN &\longrightarrow T^*N \\ (q, \dot q) &\longmapsto d(L_t|_{T_qN})_{\dot q}. \end{aligned}$$

Fact: Legendre condition $\Rightarrow$ $\mathcal{L}L_t$ invertible.

D - The Hamiltonian is the function

$$\begin{aligned} H : [0,1] \times T^*N &\longrightarrow \mathbb{R} \\ (t, q, p) &\longmapsto p\left((\mathcal{L}L_t)^{-1}(q,p)\right) - L(t, q, (\mathcal{L}L_t)^{-1}(q,p)). \end{aligned}$$

T - In local coordinates $(t, q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $[0,1] \times T^*N$ given by $p_i := \dfrac{\partial L}{\partial \dot q_i}(t, q_1, \ldots, q_n, \dot q_1, \ldots, \dot q_n)$, the E-L eqns become

$$\dot x_i = \frac{\partial H}{\partial p_i}, \quad \dot y_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n \qquad \text{(Hamilton's canonical eqns),}$$

where $y_i(t) = p_i(t, x_1(t), \ldots, x_n(t), \dot x_1(t), \ldots, \dot x_n(t))$.

Proof: In these local coordinates, $\mathcal{L}L_t(q, \dot q) = (q, p)$. In particular,

$$H = \sum_i p_i \dot q_i - L$$

Therefore,

$$\frac{\partial H}{\partial q_i} = \frac{-\partial L}{\partial q_i}, \quad \boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i}, \quad \text{and} \quad \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}.$$

Furthermore, along a solution x, the EL-eqns becomes

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} y_i = \dot{y}_i$$

$$\Rightarrow \boxed{\dot{y}_i = \frac{\partial L}{\partial q_i} = \frac{-\partial H}{\partial q_i}}.$$

We get the other implication by going backwards. ∎

Ex.: For the projectile, $\xi = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $\eta = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$, and the Hamiltonian is $H(x,y,\xi,\eta) = \frac{\xi^2}{2m} + \frac{\eta^2}{2m} + mgy$. Then, Hamilton's canonical eqns are

$$\begin{cases} \dot{x} = \frac{\xi}{m}, & \dot{\xi} = 0 \\ \dot{y} = \frac{\eta}{m}, & \dot{\eta} = -mg, \end{cases}$$

which gives the same soln (with $\dot{x}_0 = \frac{\xi}{m}$).

Rem.: When T is quadratic in the $\dot{q}_i$'s and V is independent of them, $H = T+V = E$.

2. Liouville's theorem

A Hamiltonian $H: [0,1] \times T^*N \to \mathbb{R}$ defines a flow of T^*N as follows. Let $x_0 \in T^*N$. Then, for $\varepsilon > 0$ small there exist a unique path $x: [0,\varepsilon) \to T^*N$ solving solving Hamilton's can. eqns s.t. $x(0) = x_0$ by the Picard-Lindelöf thm. If we can take ε uniformly on T^*N, then we may define

$$\varphi_H^t(x_0) := x(t) \quad \forall t < \varepsilon.$$

Thm: (Liouville) If $T^*N = \mathbb{R}^{2n}$ and $A \subseteq \mathbb{R}^{2n}$ is measurable, then $\mathrm{Vol}(\varphi_H^t(A)) = \mathrm{Vol}(A)\ \forall t$.

In fact, the thm is true whether or not H is the actual Legendre transform of a physical Lagrangian. We call $\varphi_H = \varphi_H^1$ (when $\varepsilon > 1$) a canonical transformation or Hamiltonian diffeomorphism.

1.3. Towards modern

Note that T^*N comes equipped with a canonical 2-form:

$$\omega_0 := \sum_{i=1}^{n} dp_i \wedge dq_i .$$

Hamilton's can. eqn can be written as

$$\iota_{(\dot{x},\dot{y})}\omega_0 = -dH_{(t,x,y)},$$

where $\iota_X\omega_0 := \omega_0(X,\cdot) \in \Omega^1(T^*N)$, or equivalently

$$\iota_{X_H^t}\omega_0 = -dH_t \quad , \quad \frac{d}{dt}\varphi_H^t = X_H^t \circ \varphi_H^t .$$

Observations: 1) ω_0^n is a volume form on T^*N; it is the usual one on $\mathbb{R}^{2n}$

2) $\mathcal{L}_{X_H^t}\omega_0 = d\iota_{X_H^t}\omega_0 + \iota_{X_H^t} d\omega_0^{\,0} = -d^2H_t = 0$

$\rightsquigarrow (\varphi_H^t)^*\omega_0 = \omega_0 .$

Therefore, φ_H^t preserves ω_0, and thus volume.

Rem.: If G is a Lie group which acts on T^*N through Hamiltonian diffeomorphisms ($\oplus$ technical conditions), then we may form a quotient $M = T^*N /\!\!/ G$ which inherit this 2-form. All the above then also applies to such M.

The observations implies some strong regularity on the dynamical system $(M, \{\varphi_H^t\})$. We give two examples of such phenomena.

A) Thm: Any area-preserving diffeomorphism ψ of the open disk $\{z \in \mathbb{C} \mid |z|<1\}$ has a fixed point.

B) Thm: (Gromov's nonsqueezing thm) If there exist an embedding

$$\psi : B^{2n}(r) \hookrightarrow Z^{2n}(R) := \{ q_1^2 + p_1^2 < R \}$$

s.t. $\psi^*\omega_0 = \omega_0$, then $r \le R$.

4. Hofer's geometry

Let's specialize to the case $M = \mathbb{R}^{2n}$. We want to understand

$$\mathrm{Ham}_c(\mathbb{R}^{2n}) := \{ \varphi_H \mid H \in C_c^\infty([0,1] \times \mathbb{R}^{2n}) \},$$

Fact: $\mathrm{Ham}_c(\mathbb{R}^{2n})$ is an ∞-dim. (Frechet) Lie group in the C^∞-topology with Lie algebra $(C_c^\infty([0,1]\times\mathbb{R}^{2n}), \{\cdot,\cdot\})$, where

$$\{G, H\} := \omega_0(X_H, X_G)$$

is the classical Poisson bracket.

On a finite-dim. compact Lie group G, every right-inv. norm on its Lie algebra induces a bi-inv. metric on G. Can we still have something similar on $\mathrm{Ham}_c(\mathbb{R}^{2n})$?

Hofer's observations: Every L^p-norm on $C_c^\infty(\mathbb{R}^{2n})$ is $\mathrm{Ham}_c(\mathbb{R}^{2n})$-right-inv. since Ham. diffeomorphisms pres. volume. Therefore,

$$\|\varphi\|_p := \inf_{\substack{\varphi=\varphi_H \\ H\in C_c^\infty([0,1]\times\mathbb{R}^{2n})}} \int_0^1 \|H_t\|_{L^p(\mathbb{R}^{2n})}\,dt$$

defines a conjugation-inv. pseudo-norm:

i) $\|\varphi\|_p \geq 0$ and $\|\mathbb{1}\|_p = 0$

ii) $\|\varphi\psi\|_p \leq \|\varphi\|_p + \|\psi\|_p$

iii) $\|\varphi^{-1}\|_p = \|\varphi\|_p$

iv) $\|\psi\varphi\psi^{-1}\|_p = \|\varphi\|_p$.

Rem.: 1) Physical intuition: We are minimizing the energy required to generate a Hamiltonian motion.

2) Since $\|-\|_{L^1([0,1])} \leq \|-\|_{L^p([0,1])}$ $\forall 1\leq p\leq\infty$ by Hölder's inequality, so $\int_0^1 - dt$ is the "smallest" norm possible.

Thm: [Hofer, '90] $\|-\|_p$ is

a) nondegenerate if $p=\infty$

b) $\equiv 0$ if $p<\infty$.

We set

$$\|\varphi\|_H := \|-\|_{osc}, \quad \text{where } osc(H) = \max H - \min H.$$

Then, $d_H(\varphi,\psi) := \|\varphi\psi^{-1}\|$ is a bi-inv. metric on $\mathrm{Ham}_c(\mathbb{R}^{2n})$, Idea of the proof of a): It relies on the following inequality.

Thm: [Hofer, '90] Let $\varphi \in Ham_c(\mathbb{R}^{2n})$ be s.t. $\varphi(\overline{B_r^{2n}(x)}) \cap \overline{B_r^{2n}(x)} = \emptyset$, where $x \in \mathbb{R}^{2n}$, $r > 0$. Then,

$$\|\varphi\|_H \geq \pi r^2.$$

$\square$

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