Definition A symplectic vectorspace is (V,ω) where V is a real finite dim vectorspace and $\omega : V \times V \to \mathbb{R}$ is a bilinear form such that

$$\text{(antisymmetry)} \quad \omega(u,v) = -\omega(v,u) \quad \forall u,v \in V$$

$$\text{(non-degeneracy)} \quad \omega(u,v) = 0 \ \ \forall v \in V \Rightarrow u = 0.$$

Set of such forms is $\Omega(V)$.

V is of even dimension (otherwise ω could not be non-degenerate)

Definition A linear symplectomorphism is a isomorphism $\Psi : V \to V$ such that $\Psi^* \omega = \omega$.

(recall pullback $(\Psi^*\omega)(u,v) = \omega(\Psi u, \Psi v)$).

Definition Let $W \subseteq V$ be a subspace. The symplectic complement is

$$W^\omega = \{ v \in V \mid \omega(v,w) = 0 \ \ \forall w \in W \}.$$

The subspace W is called
- isotropic if $W \subseteq W^\omega$,
- coisotropic if $W^\omega \subseteq W$,
- symplectic if $W \cap W^\omega = \{0\}$,
- Lagrangian if $W = W^\omega$.

(**Disclaimer**: When the words "shortable" or "skippable" appear in these notes it's just a note for the presentation time management and has nothing to do with math.)

Lemma $\dim W + \dim W^{\omega} = \dim V$ and $W^{\omega\omega} = W$

Proof (skippable) We define $\iota : V \to V^*$ by $\iota(v) = \omega(v, \cdot)$.
Then
$$\ker \iota = \{ v \in V \mid \omega(v, \cdot) \equiv 0 \} \underset{\omega \text{ non-degenerate}}{=} \{0\}$$

$\Rightarrow$ ι is isomorphism.

Recall the anihilator of W in V^* which is
$$W^{\perp} = \{ \varphi \in V^* \mid \varphi(w) = 0 \quad \forall w \in W \}$$

Now
$$\iota W^{\omega} = \{ \omega(v, \cdot) \mid v \in V \text{ such that } \omega(v,w) = 0 \quad \forall w \in W \}$$

$\Rightarrow$ $W^{\omega} \underset{\iota}{\cong} W^{\perp}$

Since
$$\dim W + \dim W^{\perp} = \dim V$$
the result follows. ✓

$$W^{\omega\omega} = \{ v \in V \mid \omega(v, \bar{w}) = 0 \quad \forall \bar{w} \in W^{\omega} \}$$

$\Rightarrow$ $W \subseteq W^{\omega\omega}$, $\dim W = \dim W^{\omega\omega}$ (from the first) $\Rightarrow$ $W = W^{\omega\omega}$ $\square$

Consequence W Lagrangian $\Leftrightarrow$ W isotropic and $\dim W = n$.

Theorem Let (V, ω) a symplectic vector space of $\dim = 2n$. Then there exists a **symplectic basis** $v_1, \dots, v_n, \bar{v}_1, \dots, \bar{v}_n$ which is defined to satisfy
$$\omega(v_\alpha, \bar{v}_\beta) = \delta_{\alpha\beta},$$
$$\omega(v_\alpha, v_\beta) = \omega(\bar{v}_\alpha, \bar{v}_\beta) = 0.$$

Furthermore $\exists \Psi : \mathbb{R}^{2n} \to V$ iso with $\Psi^* \omega = \omega_0$.

Proof Induction over n.

Base case: $n = 2$, take $\omega(v, u) = 1$. We have $\omega(v,v) = 0$ $\omega(u,u) = 0$.

Step: $\exists\, v_1, \bar{v}_1 \in V_1$ s.t. $\omega(v_1, \bar{v}_1) = 1$ (non-degeneracy)

$W = \operatorname{span}(v_1, \bar{v}_1)$ is symplectic (since $\omega(v, w) \neq 0$ if w, v are linear combos of $v_1, \bar{v}_1$)

Then W^ω is a symplectic vectorspace of dim $= 2n-2$.
Applying the hypo gives a symp basis $v_2, \dots, v_n, \bar{v}_2, \dots, \bar{v}_n$ of it. Then $v_1, \dots, v_n, \bar{v}_2, \dots, \bar{v}_n$ is a symp basis. ✓

For the linear map we send a vector $z \in \mathbb{R}^{2n}$ with

$$z = x^1 e_{x_1} + \dots + x^n e_{x_n} + y^1 e_{y_1} + \dots + y^n e_{y_n}$$

to

$$v = x^1 v_1 + \dots + x^n v_n + y^1 \bar{v}_1 + \dots + y^n \bar{v}_n .$$

On the example sheet we see

$$\omega_0(z, \tilde{z}) = \sum_{\alpha=1}^{n} z^{x_\alpha} \tilde{z}^{y_\alpha} - z^{y_\alpha} \tilde{z}^{x_\alpha} \qquad \text{for } z \text{ in } e_{x_i}, e_{y_i} \text{ basis}$$

$$\omega(v, \tilde{v}) = \sum_{\alpha=1}^{n} v^\alpha \tilde{\bar{v}}^\alpha - \bar{v}^\alpha \tilde{v}^\alpha \qquad \text{for } v \text{ in } v_i, \bar{v}_i \text{ basis}$$

$\Rightarrow \Psi^* \omega = \omega_0$. $\square$

Corollary If $\tilde{\omega}$ is anti-symmetric:

$$\tilde{\omega} \text{ non-degenerate} \Leftrightarrow \tilde{\omega} \wedge \dots \wedge \tilde{\omega} \neq 0.$$

Lemma $W \subseteq V$ isotropic $\Rightarrow$ $W \subseteq \Lambda$, $\Lambda \trianglelefteq V$ Lagrangian.
Every basis $u_1, \dots, u_n$ of Λ can be extended to a symplectic basis of V.

Proof (skipped) isotropic means $W \subseteq W^\omega$. Let $v \in W^\omega \setminus W$ and W_1 be $W[v]$.

$$\Rightarrow \omega|_{W_1} \equiv 0.$$

$\Rightarrow$ a maximal isotropic subspace satisfies $W = W^\omega$.

Also, $\Lambda \subseteq W \Rightarrow W^\omega \subseteq \Lambda^\omega \Rightarrow$ no subspace W that is strictly larger than a Lagrangian space Λ can be isotropic.
$\Rightarrow$ The Lagrangian subspaces are exactly the maximal isotropic subspaces. ✓

$(V,\omega) = (\mathbb{R}^{2n}, \omega_0)$, Λ Lagrangian, $J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\tilde{\Lambda} = J_0 \Lambda$ is also Lagrangian.
$\tilde{\Lambda} \cong \Lambda^*$ via ι from the previous Lemma. Choose $v_1, \dots, v_n$ dual basis to $u_1, \dots, u_n$ in $\tilde{\Lambda} \cong \Lambda^*$. ✓

$\square$

Definition The symplectic linear group is

$$Sp(V,\omega) = \{ \Psi \in GL(V) \mid \Psi^*\omega = \omega \}$$

$$Sp(2n) = Sp(\mathbb{R}^{2n}, \omega_0)$$

By the thm, it suffices to look at $Sp(2n)$.

$Sp(V,\omega)$ is a Lie group with Lie algebra
$sp(V,\omega) = \{ A \in End(V) \mid \omega(A\cdot,\cdot) + \omega(\cdot, A\cdot) = 0 \}$.

Definition A matrix A is symplectic if it satisfies
$A^T J_0 A = J_0$
where $J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. They also satisfy $\det A = 1$.

All $\Psi \in Sp(2n)$ have this property and it is equivalent to
$\Psi^*\omega_0 = \omega_0$. (see example page 5): $(\Psi^*\omega_0)(u,v) = (\Psi u)^T J_0^T \Psi v$
$= u^T \Psi^T J_0 \Psi v$.

$\mathbb{R}^{2n} \cong \mathbb{C} \rightsquigarrow J_0$ is multiplication by i
$\Rightarrow U(n) < Sp(2n)$
$GL_n(\mathbb{C}) < GL_{2n}(\mathbb{R})$

Lemma $Sp(2n) \cap O(2n) = Sp(2n) \cap GL_n(\mathbb{C}) = O(2n) \cap GL_n(\mathbb{C}) = U(n)$

In particular $sp(2n) \cap o(2n) = u(n)$.

Observation $Sp(2n) \cap O(2n)$ consists of matrices

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in GL_{2n}(\mathbb{R}), \quad X^T Y = Y^T X, \quad X^T X + Y^T Y = \mathbb{1}$$

Set $U = X + iY \Rightarrow U \in U(n)$.

Lemma $\Psi \in Sp(2n)$. Then
- $\lambda \in \sigma(\Psi) \Leftrightarrow \lambda^{-1} \in \sigma(\Psi)$ and the multiplicities agree
- If ± 1 is ev, the multiplicity is even
- $\Psi z = \lambda z$, $\Psi \tilde{z} = \tilde{\lambda} \tilde{z}$, $\lambda\tilde{\lambda} \neq 1$
$\Rightarrow \omega_0(z, \tilde{z}) = 0$

Lemma (important) $P \in Sp(2n)$ sym pos def symplectic $\Rightarrow$ $P^\alpha \in Sp(2n)$ for all $\alpha \geq 0$.

skippable Proposition $U(n)$ is a maximal compact subgroup of $Sp(2n)$.

Proposition $\iota : U(n) \hookrightarrow Sp(2n)$ is a homotopy equivalence.
So $Sp(2n)$ is connected.

skippable Proposition Every $\Psi \in Sp(2n)$ has a symplectic polar decomposition

$$\Psi = UP$$

where $U \in U(n)$ and P sym pos def symplectic.

$U = \Psi(\Psi^T\Psi)^{-\frac{1}{2}}$,

$P = (\Psi^T\Psi)^{\frac{1}{2}}$.

Proposition $\pi_1(U(n)) \cong \mathbb{Z}$.

$\det_{\mathbb{C}} : U(n) \to S^1$ induces iso of fundamental groups.

Notation $\mathcal{L}(V,\omega)$ = all Lagrangian subspaces,

$\mathcal{L}(n) = \mathcal{L}(\mathbb{R}^{2n}, \omega_0)$.

Lemma X, Y real $n\times n$ matrices. Define

$$Z: \mathbb{R}^n \to \mathbb{R}^{2n} \qquad \operatorname{im} Z = \Lambda \subseteq \mathbb{R}^{2n}$$
$$z \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} z$$

Then
$$\Lambda \in \mathcal{L}(n) \iff \operatorname{rank} Z = n \text{ and } X^T Y = Y^T X.$$

Proof (skippable) $v, \tilde{v} \in \Lambda$ are of the form $v = \begin{pmatrix} Xz \\ Yz \end{pmatrix}$ $\tilde{v} = \begin{pmatrix} X\tilde{z} \\ Y\tilde{z} \end{pmatrix}$

$$\Rightarrow \omega_0(v, \tilde{v}) = z^T (X^T Y - Y^T X)\tilde{z}$$

This is zero if and only if $X^T Y = Y^T X$.

(Since $\dim \tilde{\Lambda} = n$ $\forall \tilde{\Lambda} \in \mathcal{L}(n)$ we must have $\operatorname{rank} Z = n$.) $\square$

Corollary $\{(x, Ax) \mid x \in \mathbb{R}^n\} \in \mathcal{L}(n) \iff A$ is symmetric $\begin{pmatrix} X = \mathbb{1} \\ Y = A \end{pmatrix}$

Definition $Z = \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^{2n\times n}$ with $\operatorname{rank} Z = n$ and $X^T Y = Y^T X$ is a Lagrangian frame.

Its columns are orthonormal iff $U = X + iY \in U(n)$

(proof by seeing $Z^T Z = \mathbb{1} \iff \bar{U}^T U = \mathbb{1}$).

Then it's called unitary Lagrangian frame.

Observation $\mathcal{L}(n)$ is a manifold of $\dim = \frac{1}{2} n(n+1)$.

(See next Lemma).

Lemma

(i) $\Lambda \in \mathcal{L}(n)$ and $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ lin. sympmorpho $\Rightarrow \Psi\Lambda \in \mathcal{L}(n)$

(ii) For $\Lambda, \tilde{\Lambda} \in \mathcal{L}(n)$ $\exists \Psi \in U$ symplectic s.t. $\tilde{\Lambda} = \Psi\Lambda$.

(iii) $\mathcal{L}(n) \underset{\text{diffeo}}{\cong} U(n)/O(n)$

Proof (skippable)

(i) $(\Psi^* \omega_0) = \omega_0$. ✓

(ii) Choose a unitary frame $\binom{X}{Y}$ of Λ and define

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \qquad \Lambda_{hor} = \left\{ \binom{x}{y} \in \mathbb{R}^{2n} \,\middle|\, y = 0 \right\}$$

Then $\Psi \in Sp(2n) \cap O(2n) = U(n)$. Also $\Psi\Lambda_{hor} = \Lambda$.

Thus $\Psi' = \Psi \circ \hat{\Psi}^{-1}$ satisfies $\Psi' \tilde{\Lambda} = \Lambda$.

(iii) $U = X + iY$ is uniquely determined up to right multiplication by a matrix in $O(n)$. ✓ $\square$

Definition A linear complex structure on V is a automorphism $J: V \to V$ such that $J^2 = -\mathbb{1}$.

V with J is a complex vectorspace with scalar multiplication

$\mathbb{C} \times V \to V$

$(s+it, v) \mapsto sv + tJv$

When we identify $\mathbb{R}^{2n} \cong \mathbb{C}$ then $J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ corresponds to multiplication with i.

Proposition V $2n$ dim real vec space with complex structure J. Then there exists a isomorphism $\Phi: \mathbb{R}^{2n} \to V$ such that

$$J\Phi = \Phi J_0 .$$

Proof (shortable) Because V has a complex basis (as a complex vectorspace) $\exists\, v_1, \dots, v_n$ s.t. $v_1, \dots, v_n, Jv_1, \dots, Jv_n$ is a (real) basis of V.

Then we define
$$\Phi : \mathbb{R}^{2n} \longrightarrow V$$
$$x^\alpha e_\alpha + y^\beta \bar{e}_\beta \longmapsto x^\alpha v_\alpha + y^\beta J v_\beta$$

So $M^{(e_\alpha, \bar{e}_\alpha)}_{(v_\alpha, Jv_\alpha)}(\Phi) = \mathbb{1}$.

Then for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2n}$: $(\Phi J_0)\begin{pmatrix} x \\ y \end{pmatrix} = J_0 \begin{pmatrix} x \\ y \end{pmatrix}$ w.r.t (v_α, Jv_α)

$(J\Phi)\begin{pmatrix} x \\ y \end{pmatrix} = J\begin{pmatrix} x \\ y \end{pmatrix}$ w.r.t. (v_α, Jv_α).
$= x^\alpha J v_\alpha - y^\beta v_\beta = \begin{pmatrix} -y \\ x \end{pmatrix} = J_0 \begin{pmatrix} x \\ y \end{pmatrix}$ $\square$

Definition (V, ω) symplectic vector space. A complex structure J is compatible with ω if
$$\omega(Jv, Ju) = \omega(v, u) \quad \forall v, u \in V.$$
$$\omega(v, Jv) > 0 \quad \forall v \neq 0.$$

$\mathcal{J}(V, \omega)$ = set of all compatible complex structures

Inner product If J is compatible $g_J(u, v) = \omega(u, Jv)$ is an inner product and J is skew-adjoint
$$g_J(u, Jv) + g_J(Ju, v) = 0 \quad \forall u, v \in V.$$

Proposition (important) (V,ω) symplectic vec space with complex str. J.

TFAE

(i) $J \in \mathcal{J}(V,\omega)$

(ii) V has a symplectic basis $v_1,\dots,v_n, Jv_1,\dots,Jv_n$.

(iii) $\exists \Psi : \mathbb{R}^{2n} \to V$ isomorphism such that
$$\Psi^*\omega = \omega_0 \qquad \Psi^* J = J_0$$

(iv) $\Lambda \in \mathcal{L}(V,\omega) \Rightarrow J\Lambda \in \mathcal{L}(V,\omega)$ and $\omega(v,Jv) \neq 0 \quad \forall v \neq 0$

Proof (i) $\Rightarrow$ (ii) g_J defines an inner product. By the thm (symp basis) there exists $\Lambda \subseteq V$ Lagrangian. ($\mathrm{span}(v_1,\dots,v_n)$ is automatically Lagrangian)

Now we choose an ONB $v_1,\dots,v_n$ of Λ w.r.t. g_J.

Thus
$$\omega(v_\alpha, Jv_\beta) \overset{\text{def}}{=} g_J(v_\alpha, v_\beta) \overset{\text{ONB}}{=} \delta_{\alpha\beta}$$
and
$$\omega(Jv_\alpha, Jv_\beta) \overset{\text{comp}}{=} \omega(v_\alpha, v_\beta) \overset{\text{symp basis}}{=} 0$$
(extend $v_1,\dots,v_n$ to symplectic basis by Lemma)

$\Rightarrow v_1,\dots,v_n, Jv_1,\dots,Jv_n$ symplectic basis. ✓

(ii) $\Rightarrow$ (iii) Define
$$\Psi : \mathbb{R}^{2n} \longrightarrow V$$
$$x^\alpha e_\alpha + y^\beta \bar{e}_\beta \longmapsto x^\alpha v_\alpha + y^\beta J v_\beta$$

Then for $u = u^\alpha e_\alpha + \bar{u}^\beta \bar{e}_\beta$, $w = w^\alpha e_\alpha + \bar{w}^\beta \bar{e}_\beta$

we have
$$
\begin{aligned}
(\Psi^*\omega)(u,w) &\overset{\text{def}}{=} \omega(\Psi u, \Psi w)\\
&\overset{\text{def}}{=} \omega(u^\alpha v_\alpha + \bar{u}^\beta J v_\beta,\ w^\mu v_\mu + \bar{w}^\nu J v_\nu)\\
&\overset{\text{symp basis}}{=} u^\alpha \bar{w}^\nu \omega(v_\alpha, Jv_\nu) + \bar{u}^\beta w^\mu \omega(Jv_\beta, v_\mu)\\
&= u^\alpha \bar{w}^\beta \omega(v_\alpha, Jv_\beta) + \bar{u}^\beta w^\alpha \omega(Jv_\beta, v_\alpha)\\
&= (u^\alpha \bar{w}^\beta - \bar{u}^\beta w^\alpha)\,\delta_{\alpha\beta}\\
&= \sum_{\alpha=1}^{n} u^\alpha \bar{w}^\alpha - \bar{u}^\alpha w^\alpha\\
&= \omega_0(u,w) \quad ✓
\end{aligned}
$$
(shortable)

$$
\begin{aligned}
(\Psi^* J)(u) &\overset{\text{def}}{=} J(\Psi u)\\
&= J(u^\alpha v_\alpha + \bar{u}^\beta J v_\beta)\\
&= u^\alpha J v_\alpha - \bar{u}^\beta v_\beta\\
&= \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} u^\alpha \\ \bar{u}^\beta \end{pmatrix} = J_0(u) \quad ✓
\end{aligned}
$$

✓

(iii) $\Rightarrow$ (i) ✓

(i) $\Rightarrow$ (iv) $J \in \mathcal{J}(V,\omega)$ so by def

and $\omega(v, Jv) > 0 \quad \forall v \neq 0$ ✓

$(J^*\omega)(u,v) = \omega(Ju, Jv) = \omega(u,v) \quad \forall u,v \in V.$

Thus $\Lambda^\omega = \{ v \in V \mid \omega(v,\lambda) = 0 \quad \forall \lambda \in \Lambda \} \overset{\mathcal{L}}{=} \Lambda$

$(J\Lambda)^\omega = \{ v \in V \mid \omega(v, J\lambda) = 0 \quad \forall J\lambda \in J\Lambda \}$

this happens exactly when
$v = J\tilde{\lambda}$
since then

$\omega(J\tilde{\lambda}, J\lambda) = \omega(\tilde{\lambda}, \lambda) \overset{\Lambda \in \mathcal{L}}{=} 0$ ✓

(iv) $\Rightarrow$ (i) We define $g_J : V \times V \to \mathbb{R}$ by $g_J(u,v) = \omega(u, Jv)$. Assume by contradiction that g_J is not symmetric. Then $\exists u,v \in V$ with $\omega(v, Ju) \neq \omega(u, Jv)$. $\Rightarrow v \neq 0$. $\Rightarrow \omega(v, Jv) > 0$.

schlampig

Define $w = u - \frac{\omega(v, Ju)}{\omega(v, Jv)} v$

By straightforward computation $\omega(v, Jw) = 0$ and $\omega(w, Jv) \neq 0$.
$\Rightarrow w, Jv$ are lin. indep.
v, Jw are also lin indep. (v, u are, v, Jv too)

Since $\omega(v, Jw) = 0$ it follows by a previous Lemma (page 3) (v, Jw are contained in an isotropic subspace) that $v, Jw \in \Lambda \in \mathcal{L}(V,\omega)$.

$\Rightarrow Jv, w \in J\Lambda \in \mathcal{L}(V,\omega)$

However $\omega(Jv, w) \neq 0$ despite them being in $J\Lambda \in \mathcal{L}(V,\omega)$ ↯

Thus g_J is symmetric and hence

$\omega(u,v) = -\omega(u, J^2 v) = -\omega(u, J(Jv))$

$= -\omega(Jv, Ju) = \omega(Ju, Jv)$ ✓

$\square$

Lemma (V,ω) symplectic vectorspace of dim $2n$. Then

$$\mathcal{J}(V,\omega) \underset{\text{differ}}{\cong} \mathcal{P} = \{\text{sym pos def sympl } 2n \times 2n \text{ matrices}\}$$

In particular, $\mathcal{J}(V,\omega)$ is contractible.

Proof By the theorem assume $V = \mathbb{R}^{2n}$, $\omega = \omega_0$. $J \in \mathcal{J}(V,\omega)$ iff

- it's a complex structure $J^2 = -\mathbb{1}$

- $$\begin{aligned} \omega_0(u,v) = \omega_0(Ju,Jv) &\iff u^T J_0^T v = u^T J^T J_0^T J v \\ &\iff J^T J_0^T J = J_0^T \\ &\iff J^T J_0 J = J_0 \end{aligned}$$

- $$\begin{aligned} \omega_0(v,Jv) > 0 \quad \forall v \neq 0 &\iff v^T J_0^T J v > 0 \quad \forall v > 0 \\ &\iff \langle v, J_0^T J v\rangle > 0 \quad \forall v > 0 \\ &\iff \langle v, -J_0 J v\rangle > 0 \quad \forall v > 0. \end{aligned}$$

So, $(J_0 J)^T = J^T J_0^T = -J^T J_0 = J^T J_0 J^2 = J_0 J$

$\Rightarrow (J_0 J)^T = J_0 J \Rightarrow J_0 J$ is symmetric. Then $P = -J_0 J$ is sym pos def.

$P^T J_0 P = (J_0 J)^T J_0 (J_0 J) = J^T \underbrace{J_0^T J_0}_{\mathbb{1}} J_0 J = J^T J_0 J = J_0$

$\Rightarrow P$ is symplectic. Thus $\mathcal{J}(V,\omega) \subseteq \mathcal{P}$

If $P \in \mathcal{P}$, then $J = J_0 P \in \mathcal{J}(V,\omega)$ (similar)

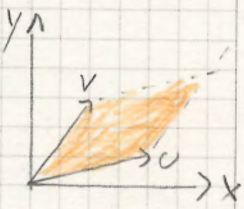
Hence $\mathcal{P} \underset{\text{diffeo}}{\cong} \mathcal{J}(V,\omega)$. By the Lemma that said $t, P \mapsto P^t$ is a homotopy from a constant map to the identity it follows that $\mathcal{P}$ is contractible. Hence so is $\mathcal{J}(V,\omega)$. $\square$

Example $(\mathbb{R}^2, \omega_0)$

Definition $\mathbb{R}^2$ with cartesian coordinates x, y and

$$\omega_0 = dx \wedge dy$$

$$\Rightarrow \quad \omega_0(u, v) = u_x v_y - u_y v_x .$$
$$= \det\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$$



Symplectomorphism

$$\Psi : \mathbb{R}^2 \longrightarrow (\mathbb{R}^2, \omega_0)$$
$$u \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} u$$

$\mathbb{R}^2$ inherits the symplectic structure from $(\mathbb{R}^2, \omega_0)$ via pullback. So, if $u, v \in \mathbb{R}^2$ then

$$\omega_{\mathbb{R}^2}(u, v) = (\Psi^* \omega_0)(u, v) = \omega_0(\Psi u, \Psi v)$$

In this case

$$\omega_0(\Psi u, \Psi v) = (-u_x)(-v_y) - (-u_y)(-v_x)$$
$$= u_x v_y - u_y v_x$$
$$= \omega_0(u, v)$$

So, $(\Psi^* \omega) = \omega_0$. $\rightsquigarrow$ Ψ is indeed a symplectomorphism.

symplectic complement

$$W = \operatorname{span}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$W^\omega = \left\{ v \in \mathbb{R}^2 \;\middle|\; \omega_0(v, w) = 0 \quad \forall w \in \operatorname{span}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$$
$$= \left\{ v \in \mathbb{R}^2 \;\middle|\; \omega_0\left(v, \lambda\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) \quad \forall \lambda \in \mathbb{R} \right\}$$
$$= \left\{ v \in \mathbb{R}^2 \;\middle|\; 3v_x\lambda - 2v_y\lambda = 0 \quad \forall \lambda \in \mathbb{R} \right\}$$
$$= \left\{ v \in \mathbb{R}^2 \;\middle|\; 3v_x - 2v_y = 0 \right\}$$
$$= \left\{ v \in \mathbb{R}^2 \;\middle|\; v_x = \tfrac{2}{3} v_y \right\}$$
$$= \operatorname{span}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

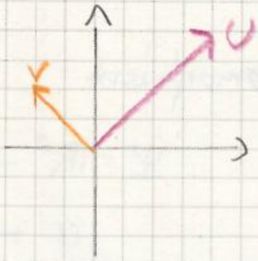
$\longrightarrow$ W is Lagrangian

$$W = \mathrm{span}\begin{pmatrix}1\\0\end{pmatrix}$$

$$W^{\omega} = \left\{ v \in \mathbb{R}^2 \;\middle|\; \omega_0\left(v, \lambda\begin{pmatrix}1\\0\end{pmatrix}\right) = 0 \quad \forall\lambda \right\}$$
$$= \left\{ v \in \mathbb{R}^2 \mid v_x 0 - v_y \lambda = 0 \quad \forall \lambda \right\}$$
$$= \mathrm{span}\begin{pmatrix}1\\0\end{pmatrix}$$

$\rightarrow$ W is Lagrangian.

Symplectic basis



$$u = \begin{pmatrix}1\\1\end{pmatrix} \qquad v = \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}$$

Check

$$\omega_0(u,u) = 0$$
$$\omega_0(v,v) = 0 \qquad \text{(antisymmetry of } \omega_0)$$

$$\omega_0(u,v) = 1 \cdot \frac{1}{2} - 1 \cdot \left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1 \qquad \checkmark$$

Linear complex structure

$$J : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad J = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \Rightarrow J^2 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

J is rotation by $\frac{\pi}{2}$. If we set $\mathbb{R}^2 \cong \mathbb{C}$ and $v \in \mathbb{R}^2 \sim z \in \mathbb{C}$ then $Jv \sim iz$.

compatibility $\quad Ju = \begin{pmatrix}-u_y\\ u_x\end{pmatrix} \quad Jv = \begin{pmatrix}-v_y\\ v_x\end{pmatrix}$

(2.5.2) $\quad dx \wedge dy(Ju, Jv) = (-u_y)v_x - u_x(-v_y) = u_x v_y - u_y v_x$
$$= dx \wedge dy(u,v)$$

(2.5.3) $\quad v \neq 0 \Rightarrow dx \wedge dy(v, Jv) = v_x v_x - v_y(-v_y) = \|v\|_2^2 > 0$

$\rightsquigarrow$ it is compatible.

corresponding inner product

$$g(u,v) = \omega(u, Jv) = u_x v_x - u_y(-v_y) = \langle u, v\rangle_2$$

Example $(\mathbb{R}^{2n}, \omega_0)$

Definition $\mathbb{R}^{2n}$ with elements $v = \begin{pmatrix} v^{x_1} \\ \vdots \\ v^{x_n} \\ v^{y_1} \\ \vdots \\ v^{y_n} \end{pmatrix}$ w.r.t. basis $e_{x_1}, \dots, e_{x_n}, e_{y_1}, \dots, e_{y_n}$.

$$\omega_0 = \sum_{\alpha=1}^{n} dx_\alpha \wedge dy_\alpha$$

general definition of such a two form

$$dx_\alpha \wedge dy_\alpha (v, V) := \det \begin{pmatrix} dx_\alpha(v) & dy_\alpha(v) \\ dx_\alpha(V) & dy_\alpha(V) \end{pmatrix}.$$

Hence

$$dx_\alpha \wedge dy_\alpha (e_{x_\mu}, e_{x_\nu}) = 0$$

$$dx_\alpha \wedge dy_\alpha (e_{y_\mu}, e_{y_\nu}) = 0$$

$$dx_\alpha \wedge dy_\alpha (e_{x_\mu}, e_{y_\nu}) = \det \begin{pmatrix} \delta_{\alpha\mu} & 0 \\ 0 & \delta_{\alpha\nu} \end{pmatrix} = \delta_{\alpha\mu}\delta_{\alpha\nu}$$

Therefore we have

$$\omega_0(v, V) = \sum_{\alpha=1}^{n} dx_\alpha \wedge dy_\alpha (v^{x_\lambda} e_{x_\lambda} + v^{y_\sigma} e_{y_\sigma}, V^{x_\mu} e_{x_\mu} + V^{y_\nu} e_{y_\nu})$$

ω_0 bilinear

$$= \sum_{\alpha=1}^{n} v^{x_\lambda} V^{y_\nu} dx_\alpha \wedge dy_\alpha (e_{x_\lambda}, e_{y_\nu}) + v^{y_\sigma} V^{x_\mu} dx_\alpha \wedge dy_\alpha (e_{y_\sigma}, e_{x_\mu})$$

relabeling dummy indices

$$= \sum_{\alpha=1}^{n} v^{x_\mu} V^{y_\nu} dx_\alpha \wedge dy_\alpha (e_{x_\mu}, e_{y_\nu}) + v^{y_\nu} V^{x_\mu} dx_\alpha \wedge dy_\alpha (e_{y_\nu}, e_{x_\mu})$$

ω_0 antisymmetric

$$= \sum_{\alpha=1}^{n} (v^{x_\mu} V^{y_\nu} - v^{y_\nu} V^{x_\mu}) dx_\alpha \wedge dy_\alpha (e_{x_\mu}, e_{y_\nu})$$

$$= \sum_{\alpha=1}^{n} (v^{x_\mu} V^{y_\nu} - v^{y_\nu} V^{x_\mu}) \delta_{\alpha\mu}\delta_{\alpha\nu}$$

$$= \sum_{\alpha=1}^{n} v^{x_\alpha} V^{y_\alpha} - v^{y_\alpha} V^{x_\alpha}$$

Result $\omega_0(v, V) = \sum_{\alpha=1}^{n} v^{x_\alpha} V^{y_\alpha} - v^{y_\alpha} V^{x_\alpha}$

(V, ω) with symplectic basis

Definition V with elements $v = \begin{pmatrix} v^1 \\ \vdots \\ v^n \\ \bar{v}^1 \\ \vdots \\ \bar{v}^n \end{pmatrix}$ w.r.t. symplectic

basis $v_1, \dots, v_n, \bar{v}_1, \dots, \bar{v}_n$.

ω is a non-degenerate skew symmetric bilinear form.

Rules based on the definition

$$\omega(v, w) = -\omega(w, v)$$

$$\omega(v_\mu, v_\nu) = 0$$
$$\omega(\bar{v}_\mu, \bar{v}_\nu) = 0$$
$$\omega(v_\mu, \bar{v}_\nu) = \delta_{\mu\nu}$$

Therefore we have

$$\omega(v, w) = \omega(v^\alpha v_\alpha + \bar{v}^\beta \bar{v}_\beta, w^\mu v_\mu + \bar{w}^\nu \bar{v}_\nu)$$
$$= v^\alpha w^\mu \omega(v_\alpha, v_\mu) + v^\alpha \bar{w}^\nu \omega(v_\alpha, \bar{v}_\nu) + \bar{v}^\beta w^\mu (\bar{v}_\beta, v_\mu) + \bar{v}^\beta \bar{w}^\nu \omega(\bar{v}_\beta, \bar{v}_\nu)$$
$$= v^\alpha \bar{w}^\nu \omega(v_\alpha, \bar{v}_\nu) + \bar{v}^\beta w^\mu \omega(\bar{v}_\beta, v_\mu)$$
$$= v^\mu \bar{w}^\nu \omega(v_\mu, \bar{v}_\nu) + \bar{v}^\nu w^\mu \omega(\bar{v}_\nu, v_\mu)$$
$$= (v^\mu \bar{w}^\nu - \bar{v}^\nu w^\mu)\, \omega(v_\mu, \bar{v}_\nu)$$
$$= (v^\mu \bar{w}^\nu - \bar{v}^\nu w^\mu)\, \delta_{\mu\nu}$$
$$= \sum_{\alpha=1}^{n} v^\alpha \bar{w}^\alpha - \bar{v}^\alpha w^\alpha$$

Result $\omega(v, w) = \sum_{\alpha=1}^{n} v^\alpha \bar{w}^\alpha - \bar{v}^\alpha w^\alpha$

P5

ω_0 written in matrix form

$$\omega_0 = \sum_{\alpha=1}^{n} U^\alpha \bar{V}^\alpha - \bar{U}^\alpha V^\alpha \quad \text{for } U = \begin{pmatrix} U^\alpha \\ \bar{U}^\alpha \end{pmatrix}, V = \begin{pmatrix} V^\alpha \\ \bar{V}^\alpha \end{pmatrix} \in \mathbb{R}^{2n}$$

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

Claim $\omega_0(U,V) = U^T J_0^T V$

Proof
$$U^T J_0^T V = (U^\alpha \quad \bar{U}^\alpha) \begin{pmatrix} 0 & +\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} V^\alpha \\ \bar{V}^\alpha \end{pmatrix}$$
$$= (U^\alpha \quad \bar{U}^\alpha) \begin{pmatrix} \bar{V}^\alpha \\ -V^\alpha \end{pmatrix}$$
$$= \sum_\alpha U^\alpha \bar{V}^\alpha - \bar{U}^\alpha V^\alpha$$
$$= \omega_0(U,V) \quad \square$$

ω_0^n as a volume form

$\mathbb{R}^2$

$$\omega_0^1 = \omega_0 = dx \wedge dy = d\,vol \quad \text{on } \mathbb{R}^2$$

$\mathbb{R}^4$

$$\begin{aligned}\omega_0^2 &= (dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \wedge (dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \\ &= dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 + dx_2 \wedge dy_2 \wedge dx_1 \wedge dy_1 \\ &= 2\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \\ &= -2\, dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2 \\ &= -2\, d\,vol\end{aligned}$$

$\mathbb{R}^{2n}$

$$\begin{aligned}\omega_0^n &= (\textstyle\sum dx_\alpha \wedge dy_\alpha) \wedge \dots \wedge (\textstyle\sum dx_\alpha \wedge dy_\alpha) \\ &= \sum_{\alpha_1 \neq \dots \neq \alpha_n = 1}^{n} dx_{\alpha_1} \wedge dy_{\alpha_1} \wedge \dots \wedge dx_n \wedge dy_n\end{aligned}$$