Reference: Differential Geometry I & II. by Prof. Will J. Merry.
(notes, 2020-2021 course)

Symplectic Manifolds

[Introduction to Symplectic Topology]:

[3.1] M: connected C^∞-smooth manifold without boundary.

Def closed manifold : compact manifold without boundary.

Def A symplectic structure on a smooth manifold M is a nondegenerate closed 2-form $\omega \in \Omega^2(M)$.

- Nondegenerate : each (T_qM, ω_q) is a symplectic vector space (only need to show it's non-degenerate. skew-symmetry is trivially satisfied if ω is 2-form).
- See P38 : M: $\dim M = 2n$. ← since $\dim T_qM$ = even. $\forall q$.
- By 21.4. $\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$, $\omega^n \in \Omega^{2n}(M)$. i.e. existence of volume form.
  $\Rightarrow$ M is oriented.

Ex $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic manifold.

$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$. where the chart : $(x_1, \cdots, x_n, y_1, \cdots, y_n)$

Note : $(\omega \wedge \theta)_p (\xi, \zeta) = \omega_p(\xi)\theta_p(\zeta) - \omega_p(\zeta)\theta_p(\xi)$. ← [DG I] 22.19

non-degeneracy : write $v = \sum a_k \frac{\partial}{\partial x_k} + \sum b_l \frac{\partial}{\partial y_l}$

$w = \sum a_k' \frac{\partial}{\partial x_k} + \sum b_l' \frac{\partial}{\partial y_l}$

$\omega_0(v, w) = \sum_{j=1}^{n} (dx_j \wedge dy_j)(v, w)$

$= \sum_{j=1}^{n} a_j b_j' - a_j' b_j$.

change w s.t. $a_j' \neq 0$, other $= 0$ vary $j = 1, \cdots, n$
into $2n$ cases and $b_j' \neq 0$ other $= 0$ vary $j = 1 \cdots n$

then we have : $a_j = b_j = 0$. for $j = 1, \cdots n$.

i.e. $v = 0$ $\Rightarrow$ non-degenerate ✓.

$\omega_0 \in \Omega^2(\mathbb{R}^{2n})$. so. skew-symmetric ✓. $d\omega = 0$ ✓ obvious

Ex (S^2, ω) is a symplectic manifold.

$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$

$\omega_x(\xi, \eta) = \langle x, \xi \times \eta \rangle$. for $\xi, \eta \in T_xS^2$.

skew-symmetric : since $\xi \times \eta$ is skew-symmetric ✓

non-degenerate : suppose $\exists \xi \neq 0$, s.t. $\omega_x(\xi, \eta) = 0$ for $\forall \eta$. ~~take η s.t. $\xi \times \eta \neq 0$.~~
~~then. since. ✗~~ since we can take η. s.t. $\xi \times \eta \neq 0$.

②

And ~~x, ξ, η~~ x and $\xi\times\eta$ are in same or inverse direction.

then $\omega_x(\xi,\eta)\neq 0$. contradiction. ✓

$d\omega=0$: see below

<u>Ex</u>. (Σ,ω) is a symplectic manifold.

where Σ is 2-dim oriented submanifold $\subseteq \mathbb{R}^3$,

equipped with $\nu:\Sigma\to S^2$ smooth s.t.

$$\nu(x)\perp T_x\Sigma.$$

$$\omega_x(\xi,\eta):=\langle \nu(x),\xi\times\eta\rangle = \det(\nu(x),\xi,\eta)$$

$$\xi,\eta\in T_x\Sigma = \nu(x)^\perp$$

<u>skew-symmetry</u>: since $\xi\times\eta$ is symmetric

<u>non-degeneracy</u>: same as above. since $\nu(x)\perp T_x\Sigma$.

~~$d\omega=0$~~ $d\omega=0$ for this <u>Ex</u> and last <u>Ex</u>:

use $d\omega(X_0,X_1,X_2)=X_0(\omega(X_1,X_2))-X_1(\omega(X_0,X_2))+X_2(\omega(X_0,X_1))$
$+(-1)^{0+1}\omega([X_0,X_1],X_2)+(-1)^{0+2}\omega([X_0,X_2],X_1)$
$+(-1)^{1+2}\omega([X_1,X_2],X_0)$.

[DG1] 23.13

to check $d\omega=0$

<u>Lemma</u>: There is a canonical isomorphism. (M,ω) sympl.

$$TM\to T^*M: \quad X\mapsto \iota(X)\omega=\omega(X,\cdot)$$

<u>Pf</u>: $(p,\xi)\mapsto(p,\omega_p(\xi,\cdot))$ ~~test~~ consider $\forall p$. $T_pM\to T_p^*M$

non-degeneracy $\Rightarrow$ injective ✓

same dim of T_pM & T_p^*M $\Rightarrow$ also bijective. ✓ ☐

<u>Rmk</u>: ~~ω is close symplectic~~ (M,ω) sympl.

ω is closed. $\Rightarrow a=[\omega]\in H^2(M;\mathbb{R})$. (see [DG1] Lecture 27)

If M is closed manifold, ~~$a^n\in H^{2n}(M;\mathbb{R})$ is represented by the volume form $\omega^n\in\Omega^{2n}(M)$~~

[DG] 27.2: $\int_{M,o}\omega^n>0$.

~~[DG] 27.3: $[\omega^n]$ generates $H^{2n}_{dR}(M)$~~ ~~$a^n=[\omega^n]$~~

[DG] 27.1: for $\omega\in\Omega_c^{m-1}(M)$ $=\Omega^{m-1}(M)$, since M is closed. $\int_{M,o}d\omega=0$.

so. we have, ω^n can not be exact ✓

~~$a^n=[\omega^n]$~~ $\Rightarrow a^n=$ ~~$[\omega^n]$~~ $a\cup\cdots\cup a=[\omega^n]\neq 0$.

[DG] 27.3. Actually. $[\omega^n]$ generates $H^{2n}_{dR}(M)$.

~~Def~~ A ~~symplectomorphism~~ of a symplectic manifold (M,ω) is a diffeo $\psi \in \mathrm{Diff}(M)$ which preserves the symplectic form

$$\omega = \psi^*\omega.$$

Denote the group of symplectomorphisms of (M,ω) by

$$\mathrm{Symp}(M,\omega) := \{\phi \in \mathrm{Diff}(M) \mid \phi^*\omega = \omega\}$$

or sometimes $\mathrm{Symp}(M)$.

Rmk. Since ω is non-degenerate, there is a one-to-one correspondence between $\mathfrak{X}(M) \to \Omega^1(M) : X \mapsto \iota(X)\omega$ (has proved before).

Def. ~~$X\in$~~ A vector field $X \in \mathfrak{X}(M)$ is symplectic if $\iota(X)\omega$ is closed.

Denote the space of symplectic vector fields by

$$\mathfrak{X}(M,\omega) := \{X \in \mathfrak{X}(M) \mid L_X\omega = d\iota(X)\omega = 0\}$$

we use $L_X\omega = d\circ\iota(X)\omega + \cancel{\iota(X)\circ d\omega}$.

$d\omega = 0$ is assumption.

Prop 3.1.5. M: closed manifold. If $t \mapsto \psi_t \in \mathrm{Diff}(M)$ is a smooth family of diffeos generated by a family of vector fields $X_t \in \mathfrak{X}(M)$, i.e.

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t \qquad \psi_0 = id$$

Then: $\psi_t \in \mathrm{Symp}(M,\omega)$, $\forall t$. if and only if $X_t \in \mathfrak{X}(M,\omega)$, $\forall t$.

Moreover, if $X, Y \in \mathfrak{X}(M,\omega)$, then $[X,Y] \in \mathfrak{X}(M,\omega)$

and $\iota([X,Y])\omega = dH$, where $H = \omega(X,Y)$

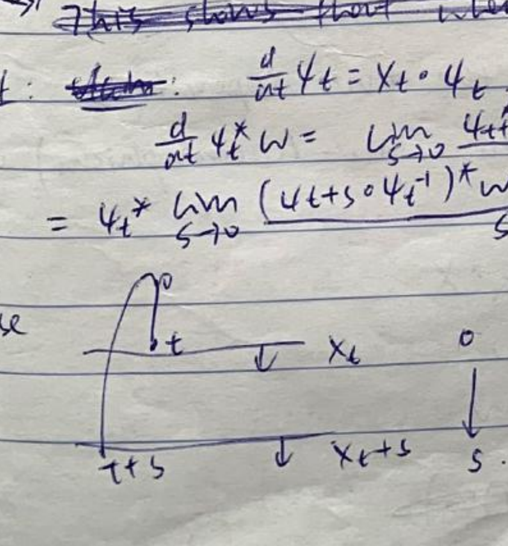
$\Rightarrow$ $\mathfrak{X}(M,\omega)$ forms a Lie algebra.

~~This shows that when M is closed, $\mathfrak{X}(M,\omega)$ is the Lie~~

Pf: $\frac{d}{dt}\psi_t = X_t \circ \psi_t$.

$$\frac{d}{dt}\psi_t^*\omega = \lim_{s\to 0}\frac{\psi_{t+s}^*\omega - \psi_t^*\omega}{s} = \lim_{s\to 0}\frac{\psi_t^*\left((\psi_t^{-1})^*(\psi_{t+s})^*\omega - \omega\right)}{s}$$

$$= \psi_t^* \lim_{s\to 0}\frac{(\psi_{t+s}\circ\psi_t^{-1})^*\omega - \omega}{s} = \psi_t^*(L_{X_t}\omega).$$

use  $\phi_s := \psi_{t+s}\circ\psi_t^{-1}$.

$$\frac{d}{dt}\phi_s = X_{t+s}\circ\phi_s.$$

④

$$L_X w = \iota(X)dw$$

$$\frac{d}{dt}\psi_t^* w = \psi_t^* L_{X_t} w \overset{\text{cartan}}{=\!=} \psi_t^*(\iota(X_t)dw + d(\iota(X_t)w))$$
$$\overset{dw=0}{=\!=} \psi_t^* (d\iota(X_t)w).$$

$$\psi_t \in \text{Symp}(M,w) \Leftrightarrow \psi_t^* w = w$$
$$\Leftrightarrow \cancel{\psi_t^*(d\iota(X_t)w)} \ \frac{d}{dt}\psi_t^* w = 0$$
$$\Leftrightarrow \psi_t^*(d\iota(X_t)w) = 0$$
$$\Leftrightarrow d\iota(X_t)w = 0$$
$$\Leftrightarrow X_t \in \mathfrak{X}(M,w). \quad \checkmark$$

Let $X, Y \in \mathfrak{X}(M,w)$. with flows ϕ_t, ψ_t.

- Sign convention in this book: (different in another book):

$$[X,Y] := -L_X Y = -\frac{d}{dt}\Big|_{t=0} \phi_t^* Y$$
$$L_X f = df \circ X = \frac{d}{dt}\Big|_{t=0} f \circ \phi_t$$
$$[L_X, L_Y] := L_X L_Y - L_Y L_X$$

with this definition, we have $L_{[X,Y]} = -[L_X, L_Y]$

$$\Rightarrow \quad \mathfrak{X}(M) \longrightarrow \text{Der}(M)$$
$$X \longmapsto L_X$$

is a Lie algebra anti-homomorphism.

$$\text{Diff}(M) \longrightarrow \text{Aut}(C^\infty(M))$$
$$\phi \longmapsto \phi^*.$$

where $\phi^*(f) := f \circ \phi$.

since $(\phi \circ \psi)^* = \psi^* \phi^*$

This map is ~~also~~ a Lie group anti-homomorphism

And differential

$$\mathfrak{X}(M) \longrightarrow \text{Der}(M)$$
$$X \longmapsto L_X$$

So. our sign convention for the two operators are consistent (both anti-homomorphism) $\checkmark$

$$[X,Y] = -[Y,X] \overset{\text{defn}}{=\!=} L_Y X$$
$$\overset{\text{defn}}{=\!=} \frac{d}{dt}\Big|_{t=0} \psi_t^* X$$

$$\iota([X,Y])w = \iota\left(\frac{d}{dt}\Big|_{t=0} \psi_t^* X\right) w$$
$$= \frac{d}{dt}\Big|_{t=0} \iota(\psi_t^* X) w$$

$$\cancel{\star \ \frac{d}{dt}\Big|_{t=0} \psi_t^* (\iota(Y) w)}$$

Claim: "☆" : $\underbrace{\iota(\psi_t^* X)}_{\text{1-form}} \omega = \psi_t^* \underbrace{(\iota(X)\omega)}_{\text{1-form}}$

Pf: $\iota(\psi_t^* X)\omega(Z) = \omega(\psi_t^* X, Z)$

$\psi_t^*(\iota(X)\omega)(Z) = (\iota(X)\omega)(D\psi_t(Z)) = \omega(X, D\psi_t(Z))$

$\forall t$, $\psi_t^*\omega = \omega$. since $\psi \in \mathfrak{X}(M,\omega)$.

$\omega(\psi_t^* X, Z)(p) = \omega_p(\underbrace{(\psi_t^* X)(p)}_{\in T_pM}, Z(p))$

[DG][21.9]: for $\lambda \in T_p^* M$

$(\psi_t^* X)_p(\lambda) = \underbrace{X_{\psi_t(p)}}_{\in T_pM} \underbrace{(D\psi_t^+(p)\lambda)}_{\in T_p^* M}$ cotangent lift.

$= (D\psi_t^+(p)\lambda)(X_{\psi_t(p)})$

$\overset{\text{By defn of cotangent lift}}{=} \lambda(D\psi_t(p)^{-1} X_{\psi_t(p)})$.

so. $(\psi_t^* X)_p = D\psi_t(p)^{-1} X_{\psi_t(p)}$.

So. $\omega(\psi_t^* X, Z)(p) = \omega_p(D\psi_t(p)^{-1} X_{\psi_t(p)}, Z(p))$

$= (\psi_t^*\omega)_p(D\psi_t(p)^{-1} X_{\psi_t(p)}, Z(p))$

$= \omega_{\psi_t(p)}(D\psi_t(p) D\psi_t(p)^{-1} X_{\psi_t(p)}, D\psi_t(p) Z(p))$

$= \omega_{\psi_t(p)}(X_{\psi_t(p)}, D\psi_t(p) Z(p))$

$= \cancel{\omega(X, D\psi_t(Z))}$ $\underline{\psi_t^*(\iota(X)\omega)(Z)(p)}$. ✓

Then use ☆. $\iota([X,Y])\omega) = \frac{d}{dt}\Big|_{t=0} \psi_t^*(\iota(X)\omega)$

$\overset{\text{By defn}}{=} L_Y(\iota(X)\omega)$

$\overset{\text{Cartan}}{=} d\circ\iota(Y)\iota(X)\omega + \iota(Y)\circ \cancel{d\iota(X)}\omega$

$\overset{X\in \mathfrak{X}(M,\omega)}{=} d\iota(Y)\iota(X)\omega$

$= d\omega(X,Y)$ ✓.

$\iota(Y)\iota(X)\omega = \underbrace{\iota(X)\omega}_{\text{1-form}}(Y) = \omega(X,Y)$. ☒

Def. For any smooth function $H: M \to \mathbb{R}$. The vector field $X_H: M \to TM$ determined by the identity

$\iota(X_H)\omega = dH$

is called the Hamiltonian vector field associated to the Hamiltonian function H.

~~Rmk~~ Def^n

If M is closed, ~~the~~ By [DG1] 9.19. every vector field on M is complete, i.e. $\mathbb{R} \to \mathrm{Diff}(M)$
domain of the flow.

so. X_H generates a smooth 1-parameter group of diffeos $\phi_H^t \in \mathrm{Diff}(M)$ satisfying

$$\frac{d}{dt}\phi_H^t = X_H \circ \phi_H^t, \quad \phi_H^0 = id$$

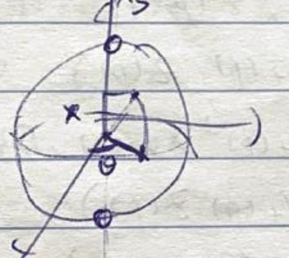
This is called the Hamiltonian flow associated to H.

we have. $dH(X_H) \underset{\text{By def}^n}{=} \iota(X_H)\omega(X_H) = \omega(X_H, X_H) = 0.$

This shows. X_H is tangent to the level sets $H = $ const of H.

EX. cylindrical polar coordinates (θ, x_3) on $S^2 \setminus \{(0,0,\pm 1)\}$.

$0 \le \theta < 2\pi$, $-1 < x_3 < 1$



area form: $\omega = d\theta \wedge dx_3$.

take $H: S^2 \to \mathbb{R}$ be. $H = x_3$.

use $\iota_{X_H}(H) = 0$. and $\omega = d\theta \wedge dx_3$.

we get $X_H = \frac{\partial}{\partial\theta}$.

$\Rightarrow$ flow: ϕ_H^t is the rotation of the sphere about its vertical axis through the angle t.

Def ~~A smooth function F~~ Poisson bracket:

$$\{F, H\} := \omega(X_F, X_H) = dF(X_H). \quad F, H \in C^\infty(M)$$

Then, a smooth function $F \in C^\infty(M)$ is constant along the orbits of the flow of H if and only if $\{F, H\} = 0$.

~~Thm~~ Rmk: Poisson bracket defines a Lie algebra structure on $C^\infty(M)$

Pf: 1.1.18: shows ω_0 on $C^\infty(\mathbb{R}^{2n})$ leads to Lie algebra ~~de~~ formed by $\{\cdot,\cdot\}$

Darboux's thm. shows ω on M locally diffeo to ω_0 on $\mathbb{R}^{2n}$.

will show later.

∎

prop 3.1.10 (M,ω) sympl.

(1). Whenever defined, the Hamiltonian flow ϕ_H^t is a symplectomorphism, which is tangent to the level sets of H.

(2) For every Hamiltonian function $H: M \to \mathbb{R}$ and every symplectomorphism $\psi \in Symp(M,\omega)$, we have $X_{H\circ\psi} = \psi^* X_H$.

(3) The Lie bracket of two Hamiltonian vector fields X_F and X_G is the Hamiltonian vector field

$$[X_F, X_G] = X_{\{F,G\}}$$

Pf (1). By 3.1.5 $\phi_H^t \in Symp(M) \Rightarrow X_H \in \mathcal{X}(M,\omega)$ i.e. $\iota(X_H)\omega = dH$.

$$X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0.$$

$\Rightarrow X_H$ & ϕ_H^t are tangent to the level surface of H. ✓

(2). $\iota(X_{H\circ\psi})\omega \overset{def^n}{=} d(H\circ\psi) \overset{(DG)\,23.4}{=} \psi^*(dH) \overset{def^n}{=} \psi^*\iota(X_H)\omega = \iota(\psi^* X_H)\omega$

$\psi^* d\omega = d\psi^*\omega$

(DG) 21.8: $\psi^* f = f\circ\psi$

$\psi \in Symp(M)$. use the same method as "☆" in 3.1.5

$\omega(X_{H\circ\psi}, \cdot) = \omega(\psi^* X_H, \cdot)$

Since ω is non-degenerate $\Rightarrow X_{H\circ\psi} = \psi^* X_H$. ✓

(3) $X_F, X_G \in \mathcal{X}(M,\omega) \Rightarrow \phi_F^t \in Symp(M,\omega)$

$$[X_F, X_G] \underset{\text{sign convention}}{=} -\frac{d}{dt}\Big|_{t=0} (\phi_F^t)^* X_G \underset{\substack{\text{use}\\(2)}}{=} -\frac{d}{dt}\Big|_{t=0} \cancel{X_{G\circ\phi_F^t}}\, X_{G\circ\phi_F^t}$$

$$\begin{aligned}
\iota([X_F, X_G]\omega) &= \iota\Big(-\frac{d}{dt}\Big|_{t=0} \overset{X_{G\circ\phi_F^t}}{\cancel{X_{G\circ\phi_F^t}}}\Big)\omega \\
&= -\frac{d}{dt}\Big|_{t=0} \underbrace{\iota\,(\cancel{X_{G\circ\phi_F^t}})\,(X_{G\circ\phi_F^t})\,\omega} && \text{By def}^n \text{ of symplectic vector field.} \\
&= -\frac{d}{dt}\Big|_{t=0} \underline{d(G\circ\phi_F^t)} \\
&= -d\,\underbrace{\frac{d}{dt}\Big|_{t=0} G\circ\phi_F^t} && \text{By def}^n \\
&= -d(\underbrace{X_F(G)}) \\
&= -d(dG(X_F)) \\
&= -d\{G,F\} && \text{By def}^n \\
&= d\{F,G\} && \text{since } \omega \text{ is skew-symmetric } \& \text{ def}^n \text{ of } \{\cdot,\cdot\}
\end{aligned}$$

$\Rightarrow X_{\{F,G\}} = [X_F, X_G]$ ✓

Rmk: By 3.1.10 (3), the Hamiltonian vector fields form a Lie subalgebra of symplectic vector fields

$(\iota(X_F)\omega = dF)$,

By defn of Hamiltonian vector fields. $H \mapsto X_H$ is a surjective Lie algebra homomorphism:

$$C^\infty(M) \longrightarrow \mathcal{X}(M,\omega)$$
$$\{\cdot,\cdot\} \qquad [\cdot,\cdot]$$

The kernel of this homomorphism consists of constant functions: Pf: $X_H = 0 \Rightarrow \iota(X_H)\omega = dH = 0 \Rightarrow H$ is const ✓

Rmk (M,ω) compact connected symplectic manifold.

Ex 3.1.9 shows: $\int_M \{F,G\}\omega^n = 0$, if (M,ω) sympl. and one of F, G has ~~constan~~ compact support.

Use Ex 3.1.9 (since M compact), we have: the space of smooth functions $F: M \to \mathbb{R}$ that have mean value zero with respect to ω (i.e. $\int_M F\omega^n = 0$) is closed under $\{\cdot,\cdot\}$. ✓

- $H \mapsto X_H$ is a Lie algebra isomorphism from $C^\infty(M) \cap \{\text{mean value zero}\} \longrightarrow \mathcal{X}(M,\omega)$
$$\{\cdot,\cdot\} \qquad [\cdot,\cdot]$$

Pf: Last remark has shown: surjective, since changing the mean value doesn't impact on the vector field. < Also, $\{\cdot,\cdot\} \to [\cdot,\cdot]$ is homomorphism

Only need to show: injective.

$F \to X_F$, $G \to$. $\iota(X_F) = dF = dG \Rightarrow d(F-G) = 0$
$\Rightarrow F - G$ const.

But $\int_M F\omega^n = \int_M G\omega^n = 0 = \int_M (F-G)\omega^n$

~~so $F-G=0$~~. And $\int_M \omega^n > 0$.

~~then~~ $F - G$ must be $0 \Rightarrow$ injective ✓ ∎

Rmk ~~By 3.1~~

(1) By 3.1.10 (1), the Hamiltonian function H is constant along the flow lines of the associated Hamiltonian vector field X_H. ✓

Hence every level set of H is an invariant submanifold of M.

flow

$\downarrow \forall x_0 \in S.$

Def^n of invariant set: $\forall\ t \longmapsto \phi_t(x_0)$ defined on it's maximal interval of existence has it's image in S

invariant manifold: if S in addition is a manifold.

(2) conversely, let $S \subset M$ be any compact oriented hypersurface (that is, a submanifold of ~~a~~ codim 1) of a symplectic manifold (M, ω). By 2.1.14, each such hypersurface is a coisotropic submanifold: (look at each T_pS).

coisotropic

$L_q := (T_qS)^{\omega} \overset{P38}{=} \{v \in T_qM \mid \omega(v, w) = 0 \text{ for } w \in T_pS\} \subseteq T_qS$

is a 1-dim subspace of T_qS for $\forall q \in S$.

$\uparrow$ use 2.1.1. $\dim T_qS + \dim L_q = \dim T_qM$.

since S is codim 1 $\Rightarrow \dim L_q = 1$. ✓

$\Delta := \coprod_{q \in S} L_q$ defines a distribution on S

[DG] 14.6: for one-dim case, integral manifold always exists around every point.

[DG] 14.11: since at every point, $\exists$ an integral manifold around p, $\Rightarrow \Delta$ is integrable.

[DG] 15.4: ~~induced~~ Δ is induced by a foliation

i.e. It integrates to give a 1-dim foliation of S, called the characteristic foliation.

- The leaves of this foliation are the integral curves of any Hamiltonian vector field X_H for which S is a regular level surface of the function H. (or a component of such a surface).

Pf: Need to show: ~~$\forall q$~~ Given H.

$\forall q \in S$. $\phi_H^t(q) \in S$. for $t \in (a,b)$ - domain of ϕ_H^t.

by def^n of ~~E~~ Δ

i.e. $\forall t$. $\forall v \in T_{\phi_H^t(q)}S$. $\omega(X_H(\phi_H^t(q)), v) = 0$.

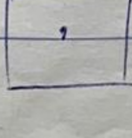
~~$\omega(\phi_H^t(q))$~~

$\quad \| \quad$ def^n of ~~a~~ sympl. vector field.

$(dH)_{\phi_H^t(q)}(v)$

$\quad \| $

0. since ~~$\phi_H^t(q)$~~ ϕ_H^t is tangent to ~~level~~ level sets of H. ☒



$T_{\phi_H^t(q)}S$

Def. (M,ω) sympl. manifold without boundary.
A symplectic isotopy of (M,ω) is a smooth map
$[0,1]\times M \to M$. such that ψ_t is a ~~symplectic~~
$(t,q) \mapsto \psi_t(q)$
symplectomorphism for every t and ψ_0 is the identity.

Any such isotopy is generated by a smooth family of vector fields $X_t : M \to TM$ via

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = id.$$

Since $\psi_t \in Symp(M)$, $\forall t$. $\Rightarrow X_t \in \mathcal{X}(M,\omega)$, $\forall t$.

A symplectic isotopy $\{\psi_t\}_{0\le t\le 1}$ is called a Hamiltonian isotopy if the 1-form $\iota(X_t)\omega$ is exact.
~~and s~~ By def^n. X_t is a Hamiltonian vector field, $\forall t$.
In this case, there is a smooth function $H:[0,1]\times M \to \mathbb{R}$ s.t. $\forall t$. $H_t := H(t,\cdot)$ generates the vector field X_t via $\iota(X_t)\omega = dH_t$.

H is called a time-dependent Hamiltonian.
It is determined by the Hamiltonian isotopy only up to an additive function $c:[0,1]\to\mathbb{R}$
(i.e. $\forall t$. c_t is a const map.
so. ~~rega~~ c is regarded as $c:[0,1]\to\mathbb{R}$)

- If M is simply connected, then $H^1_{dR}(M)=0$.
1-closed form is exact. then every symplectic isotopy is a Hamiltonian isotopy.

Def. A symplectomorphism $\psi \in Symp(M,\omega)$ is called Hamiltonian if there exists a Hamiltonian isotopy $\{\psi_t \in Symp(M,\omega)\}$ from $\psi_0 = id$ to $\psi_1 = \psi$.
Denote the space of Hamiltonian symplectomorphisms by

$$\mathrm{Ham}(M,\omega) := \left\{ \psi \in \mathrm{Symp}(M,\omega) \;\middle|\; \begin{array}{l} \exists\, [0,1] \to C^\infty(M)\ t \mapsto H_t \\ \exists\, [0,1] \to \mathrm{Diff}(M) : t \mapsto \psi_t \\ \text{with } \boxed{\begin{array}{l} \frac{d}{dt}\psi_t = X_t \circ \psi_t \,.\ \psi_0 = id \\ \iota(X_t)\omega = dH_t \end{array}} \;\star\star \\ \psi_1 = \psi \end{array} \right\}.$$

Every compactly supported Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$ determines a compactly supported Hamiltonian isotopy. $\{\psi_t\}_{0 \le t \le 1}$ via $\star\star$.

And its time-1 map is denoted by $\phi_H := \psi_1$

Every such Hamiltonian symplectomorphism is called <u>compactly supported</u>, and we denote

$$\mathrm{Ham}_c(M,\omega) := \{ \phi_H \mid H \in C_0^\infty([0,1] \times M) \}$$ (with compact support)

When M is closed. $\mathrm{Ham}_c(M,\omega) = \mathrm{Ham}(M,\omega)$.

Also ~~we~~ sometimes write $\mathrm{Ham}(M) := \mathrm{Ham}(M,\omega)$

$\mathrm{Ham}_c(M) := \mathrm{Ham}_c(M,\omega)$.

- <u>Fact</u>: (Ex 3.1.14). $\mathrm{Ham}(M,\omega)$ is a normal subgrp of $\mathrm{Symp}(M,\omega)$
  And its Lie algebra is the algebra of all Hamiltonian vector fields

Rmk In closed manifolds, ~~there is~~

[DG] 9.15 : 1-1 correspondence : one-parameter subgroups & complete vector fields.

[DG] 9.19 : If M is compact. $\forall X \in \mathfrak{X}(M)$. X is complete.

$\Rightarrow$ 1-1 correspondence : ~~isotopies~~ flows and ~~time-dependent~~ vector fields

- - - - : isotopies and time-dependent vector fields
- - - - : symplectic isotopies and time-dep. sympl. vector fields
- - - - : Hamiltonian isotopies and time-dep. Hamiltonian vector fields

For non-closed manifolds, it's complicated.

Ex Symplectic manifolds

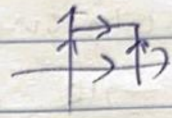
1. Every oriented Riemann surface with its area form is a symplectic manifold.

closed: $\dim = 2 \Rightarrow d\omega = 0$ for $\omega \in \Omega^2(M)$.

Pf: non-degenerate: since volume form is nowhere -vanishing.

skew-symmetric: trivial.

② $2n$-dim ~~T~~ torus $\mathbb{T}^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n}$

with its standard form 

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$

closed ✓ skew-symmetric ✓ non-degenerate: similar method to ω_0 in $\mathbb{R}^{2n}$.

③ product of two symplectic manifolds

$M_1 \times M_2$:

with symplectic form $\omega_1 \oplus \omega_2 := pr_1^*\omega_1 + pr_2^*\omega_2$.

[OG] 23.4: φ is smooth $\Rightarrow \varphi^*(d\omega) = d(\varphi^*\omega)$

Hence $\omega_1 \oplus \omega_2$ is also closed. ✓

non-degenerate: use $T_{(p,q)} M_1 \times M_2 \cong T_p M_1 \oplus T_q M_2$

~~suppose $v_1 \neq 0$, $v_2 \neq 0$~~ Also, since ω_1, ω_2 both non-degenerate, $\Rightarrow$ $\omega_1 \oplus \omega_2$ must be also non-degenerate.

(can suppose $\exists\, V \neq 0$ st. $\omega_1 \oplus \omega_2 (V, \cdot) = 0$.

Then decomposite V into $v_1 + v_2$.

Then $v_1 = 0$, $v_2 = 0$ $\Rightarrow\Leftarrow$ (can take $\bar{w}_1 = 0$, $\bar{w}_2 \neq 0$ and $\bar{w}_1 \neq 0$ $\bar{w}_2 = 0$ ~~to get this~~ $\omega_1 \oplus \omega_2 (V, \bar{w}) = 0$ to get $v_1 = 0$, $v_2 = 0$) ✓

skew-symmetric: trivial ✓

Ex (A 4-dim symplectic manifold)

Consider the group $\Gamma = \mathbb{Z}^2 \times \mathbb{Z}^2$ with the noncommutative group action $(j', k') \circ (j, k) = (j + j', A_{j'} k + k')$,

$A_j = \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix}$. $j = (j_1, j_2) \in \mathbb{Z}^2$. $k = (k_1, k_2) \in \mathbb{Z}^2$

Γ acts on $\mathbb{R}^4$ via

$\Gamma \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$: $((j,k), (x,y)) \mapsto (x + j, A_j y + k)$

This action preserves the symplectic form $\omega := dx_1 \wedge dx_2 + dy_1 \wedge dy_2$.

$M = \mathbb{R}^4/\Gamma$ is a compact symplectic manifold.

We use a result, which says: X: simply connected top. space.
G: group acts on X. $\forall x \in X$. $\exists$ nbhd U s.t. $\forall g \in G$, $gU \cap U = \phi$.
unless g is the identity, then $\pi_1(X/G) = G$.

Then, $\pi_1(M) = \Gamma$.

By Hurewicz Thm, $H_1(M;\mathbb{Z}) = \underbrace{\Gamma/[\Gamma,\Gamma]}_{= \pi_1(M)/[\pi_1(M),\pi_1(M)]} \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$.

where $[\Gamma,\Gamma] = 0\oplus 0\oplus \mathbb{Z}\oplus 0$ is the ~~sub~~ group of commutators.

Since the odd-dimensional Betti numbers $b_k(X) = \operatorname{rank} H_k(X)$ of a compact Kähler manifold must be even, but here $b_1(M) = 3$, then, M does not admit a Kähler structure.

Consider σ: $\overset{=G}{\mathbb{Z}^2}$ acts on $\overset{=L}{\mathbb{T}^2}$ via $j \longmapsto A_j$

claim: $\underset{P}{\overset{\pi:\ \mathbb{R}^2}{\shortparallel}} \longrightarrow \underset{\mathbb{T}^2}{\overset{M}{\shortparallel}}$ is a principal $\mathbb{Z}^2$-bundle.

$\left(\begin{array}{l}\text{Then. } P\times_G L \to M \text{ is also a bundle. (see [DG])} \\ M = \mathbb{T}^2 = L.\end{array}\right)$

Pf: surjective submersion ✓. since covering map.

①. τ: $\mathbb{Z}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$((a,b),(x,y)) \longmapsto (x+a, y+b)$ is smooth action

free. $\exists (x,y)$ s.t. $(x,y) = (x+a, y+b)$

$\Rightarrow (a,b) = (0,0)$ ✓

② fibre-preserving: $\pi: \mathbb{R}^2 \longrightarrow \mathbb{T}^2$.

for $p = (x,y) \in \mathbb{T}^2$ $\quad P_p = \pi^{-1}(p) = (x+\mathbb{Z}, y+\mathbb{Z})$ for $(x,y)\in[0,1)^2$.

$\forall u \in (x+a_1, y+b_1) \in P_p$. $\tau_{(a,b)}(u) = (x+a_1+a, y+b_1+b) \in P_p$ ✓.

③ transitive on the fibres:

since $(a,b)\in\mathbb{Z}^2$. $P_p = (x+\mathbb{Z}, y+\mathbb{Z})$.

$\Rightarrow$ $\mathbb{Z}^2$ acts on P_p transitively. ✓

Then. $\pi: \mathbb{R}^2 \longrightarrow \mathbb{T}^2$ is a principal $\mathbb{Z}^2$-bundle. ✓

Then, $\boxed{\mathbb{R}^2\times_{\mathbb{Z}^2}\mathbb{T}^2 \text{ is an associated bundle of } \mathbb{T}^2.}$

π_L: $\underset{P}{\mathbb{R}^2} \times_{\mathbb{Z}^2} \underset{L}{\mathbb{T}^2} \longrightarrow \mathbb{T}^2$ is a $(\mathbb{Z}^2,\sigma)$-fibre bundle.

$[u, q] \longmapsto \pi(u)$

(12+2)

the equivalence relation is $(\tau g(u), q) \sim (u, \sigma g(q))$.

So. $((x+a, y+b), (p, q)) \sim ((x, y), \underbrace{p+jq, q}_{\text{mod } \mathbb{Z}^2})$ $(\star)$

Consider $M = \mathbb{R}^4/T$. compare. M with $\mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{T}^2$

$T \times \mathbb{R}^4 \to \mathbb{R}^4$.

$(j, k) \quad ((x, y)) \longmapsto ((x+j), A_j y + k)$

$$A_j y + k = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$= (y_1 + j y_2 + k_1, y_2 + k_2)$$

since $(k_1, k_2) \in \mathbb{Z}^2$, if we look inside $\mathbb{T}^2$, we get

(y_1, y_2) corresponds to $(y_1 + j y_2, y_2)$.

which is just the same as "(p, q) corresponds to $(p+jq, q)$".

Hence. $M = \mathbb{R}^4/T \;=\!=\; \mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{T}^2$ ✓

~~Also~~, Claim: Also, $M = [0,1] \times S^1 \times \mathbb{T}^2/\sim$.

where $(0, x_2, y_1, y_2) \sim (1, x_2, y_1 + y_2, y_2)$

Pf: Since $[0,1]$ with $0 \sim 1$ is homeo to S^1.

$\mathbb{T}^2 \cong S^1 \times S^1$.

and also compare this equivalence with $(\star)$.

we can get that this is true. ✓ $\boxtimes$

Ex cotangent bundles

T^*L is the vector bundle whose sections are 1-forms on L

~~and so it carries a universal~~

Claim: (T^*L, ω_{can}) is a sympletic manifold.

where $\omega_{can} = -d\lambda_{can}. \in \Omega^2(T^*L)$

$\lambda_{can} \in \Omega^1(T^*L)$.

In standard local coordinates (x,y). where $x \in \mathbb{R}^n$ is the coordinate on L and $y \in \mathbb{R}^n$ is the coordinate on the fibre T_xL.

$\lambda_{can} := y\,dx.$ $\quad \omega_{can} = -d\lambda_{can} = -dy \wedge dx = dx \wedge dy.$

Pf: Let $x: U \to \mathbb{R}^n$ be a local coordinate chart on L.

~~$\forall v^* \in$~~ $q \in U$. $\forall v^* \in T_q^*L$

$v^* = \sum_{j=1}^n y_j\,dx_j.$

The coordinates y_j are uniquely determined by q and v^*, and determine coordinate functions

$y: T^*U \to \mathbb{R}^n$
$(q, v^*) \mapsto y(q, v^*)$.

In summary, we have a coordinate chart $T^*U \to \mathbb{R}^n \times \mathbb{R}^n$
$(q, v^*) \mapsto (x(q), y(q, v^*))$

to prove it's a chart. need to show bijection: surjective: obvious. injective: x ✓ y: ✓

In these coordinates the canonical 1-form is given by

$$\boxed{\lambda_{can} := \sum_{j=1}^n y_j\,dx_j.} \qquad y_j: T^*U \to \mathbb{R}.$$

~~Another way~~ Consider the projection

$$\pi: T^*L \to L$$
$$(q, v^*) \mapsto q.$$

$$d\pi(q, v^*): T_{(q,v^*)}T^*L \to T_qL$$
$$(\xi, \eta) \mapsto \xi.$$

notice that
$d\pi(q,v^*)\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}$
$d\pi(q,v^*)\frac{\partial}{\partial y_j} = 0$

since π is a projection.

$$v^* \circ d\pi(q, v^*): (\xi, \eta) \mapsto \xi \mapsto v^*(\xi). = \left(\sum_{j=1}^n y_j\,dx_j\right)(\xi)$$

so. $\boxed{\lambda_{can}{}_{(q,v^*)} = v^* \circ d\pi(q, v^*)}$, since η is with $\frac{\partial}{\partial y_j}$ and $v^*(\eta) = 0$. ✓

use local coordinate to calculate, $\omega_{can} = -d\lambda_{can}$ is non-degenerate ✓
$\Rightarrow (T^*L, \omega_{can})$ is sympl. mfd ∎

prop. 3.1.18 The 1-form $\lambda_{can} \in \Omega^1(T^*L)$ is uniquely characterized by the property that $\sigma^* \lambda_{can} = \sigma$

for every 1-form $\sigma : L \to T^*L$

Pf : In the local coordinates x on L, a 1-form σ on L can be written as $\sigma = \sum_{j=1}^{n} a_j(x) dx_j$.

Then $L \xrightarrow{\sigma} T^*L$

$$x = (x_1, \cdots, x_n) \longmapsto \underbrace{(x_1, \cdots, x_n, a_1(x), \cdots, a_n(x))}_{"=(x,y)"}$$

So. in local coordinates, to calulate $\sigma^* \lambda_{can}$, we just need to change "y_j" in λ_{can}. into "$a_j \circ x$".

i.e. $\sigma^* \lambda_{can} = \sum_{j=1}^{n} a_j \circ x \, dx_j = \sigma$. ▨

$\boxed{3.2}$

~~Rmk~~ Fact ① : Moser's argument shows that, for every family of symplectic forms $\omega_t \in \Omega^2(M)$ with an exact derivative. $\frac{d}{dt}\omega_t = d\sigma_t$.

there exists a family of diffeos $\psi_t \in \mathrm{Diff}(M)$ s.t.

$$\psi_t^* \omega_t = \omega_0.$$

② The key idea is to determine the diffeomorphisms ψ_t by representing them as the flow of a family of vector fields X_t on M.

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = id.$$

The vector fields X_t has to be constructed s.t. $\psi_t^* \omega_t = \omega_0$ is satisfied.

Rmk: Differentiate this.

$$\frac{d}{dt}\psi_t^* \omega_t = \lim_{s \to 0} \frac{\psi_{t+s}^* \omega_{t+s} - \psi_t^* \omega_t}{s}$$

$$= \lim_{s \to 0} \psi_t^* \frac{(\psi_{t+s} \circ \psi_t^{-1})^* \omega_{t+s} - \omega_t}{s}$$

$$= \psi_t^* \lim_{s \to 0} \frac{(\psi_{t+s} \circ \psi_t^{-1})^* \omega_{t+s} - \omega_t}{s}$$

$\omega_{t+s} = \omega_t + s \cdot \frac{d}{dt}\omega_t + ~~O(s^2)~~ \, o(s)$

$$\to = \psi_t^* \left(\lim_{s \to 0} \frac{(\psi_{t+s} \circ \psi_t^{-1})^* \omega_t - \omega_t}{s} + \lim_{s \to 0} (\psi_{t+s} \circ \psi_t^{-1})^* \frac{d}{dt}\omega_t + \frac{o(s)}{s} \right).$$

$$= \psi_t^* (\mathcal{L}_{X_t} \omega_t) + \psi_t^* \frac{d}{dt}\omega_t.$$

Cartan $\psi_t^*\left(\frac{d}{dt}\omega_t + \iota(X_t)\underbrace{d\omega_t}_{\text{closed}, \forall t} + d\iota(X_t)\omega_t\right)$

$= \psi_t^*\left(\frac{d\sigma_t}{dt} + d\iota(X_t)\omega_t\right) = \psi_t^*\left(d\left(\sigma_t + \iota(X_t)\omega_t\right)\right)$

~~If we want $\psi_t^*\omega_t = \omega_0$, then ω_0~~

then, if we can show $\sigma_t + d\iota(X_t)\omega_t = 0$

then we can get $\frac{d}{dt}\psi_t^*\omega_t = 0 \Rightarrow \psi_t^*\omega_t = \omega_0$.

Lemma 3.2.1 (Moser isotopy)

Let M be a $2n$-dim smooth manifold. $Q\subset M$ compact submanifold. $\omega_0, \omega_1 \in \Omega^2(M)$ are closed 2-forms s.t. at each point q of Q, the forms ω_0 and ω_1 are equal and non-degenerate on T_qM.

Then, there exists open nbhds N_0 and N_1 of Q and a diffeomorphism $\psi: N_0 \to N_1$ s.t. $\psi|_Q = id$, $\psi^*\omega_1 = \omega_0$

Pf: Use Moser's argument.

It's enough to prove: $\boxed{\exists \text{ 1-form } \sigma \in \Omega^1(N_0) \text{ s.t. } \sigma|_{T_QM} = 0,\ d\sigma = \omega_1 - \omega_0}$ ☆☆☆

Then, we consider ~~the~~ the family of closed forms

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + t\,d\sigma \quad \text{on } N_0.$$

$d\sigma$ is closed, since ω_0, ω_1 closed.

Shrinking N_0 if necessary, w.m.a. ω_t is non-degenerate in N_0, $\forall t$. ⟸ this can be achieved, since ω_0, ω_1 non-degenerate on T_qM, $\forall q \in Q$.

Shrinking N_0 if necessary, w.m.a. the solutions of $\psi_t^*\omega_t = \omega_0$ exists on $0 \le t \le 1$. ⟸ this can be achieved, since $\omega_0 = \omega_1$ on T_qM, $\forall q \in Q$

Pf of ☆☆☆:

consider the restriction of the exponential map to the normal bundle $TQ^\perp$ of the submanifold Q with respect to any Riemannian metric on M.

$\exp: TQ^\perp \to M$.

(16)

Consider the nbhd of the zero section $\{(q,u) \in TM \mid q \in Q,\ u=0\}$

$$U_\varepsilon := \{(q,v) \in TM \mid q \in Q,\ v \in T_qQ^\perp,\ |v|<\varepsilon\}$$

Then $U_\varepsilon \xrightarrow{\exp} N_0 := \exp(U_\varepsilon)$ is diffeo

for $\varepsilon > 0$ sufficiently small.

Define $\phi_t : N_0 \to N_0$ for $0 \le t \le 1$ by

$$\phi_t(\exp(q,v)) := \exp(q,tv)$$

for $t>0$. ϕ_t is a diffeo inside N_0.

And we have $\phi_0(N_0) \subset Q$.

$$\phi_1 = id.$$

since $\exp(p,0_p)=p \to \phi_t|_Q = \phi_t|_{Q\times\{0\}} = id.$

~~This implies~~ $\cancel{\phi_0^* \tau =}$ Denote $\tau := \omega_1 - \omega_0$.

This implies $\phi_0^*\tau = 0$. $\phi_1^*\tau = \tau$.

~~Since~~ since $\tau = 0$ on Q

and $\phi_0(N_0) \subset Q$. ✓

Since ϕ_t is a diffeo for $t>0$, we defined.

$$X_t := \left(\frac{d}{dt}\phi_t\right)\circ \phi_t^{-1} \quad \text{for } t>0$$

Define $\sigma_t := \phi_t^*(\iota(X_t)\tau)$.

for $w \in T_xN_0$. $\sigma_t(x;w) = \phi_t^*(\iota(X_t)\tau)(x;w)$

$$= (\iota(X_t)\tau)_{\phi_t(x)}(D\phi_t(x)w)$$

$$= \tau_{\phi_t(x)}(X_t(\phi_t(x)),\ D\phi_t(x)w).$$

since $X_t \circ \phi_t := \frac{d}{dt}\phi_t$ for $t>0$

consider $\lim_{t\to0} X_t\circ\phi_t(\exp(q,v))$

$$= \lim_{t\to0}\frac{d}{dt}\phi_t(\exp(q,v))$$

$$= \frac{d}{dt}\Big|_{t=0}(\phi_t\circ\exp(q,v))$$

$$= \frac{d}{dt}\Big|_{t=0}\exp(q,tv)$$

$$= D\exp_q(0_q)\circ J_{0_q}(v)$$

$$= id(v)$$

$$= v. \qquad \text{then } \sigma_t \text{ is well-defined at } t=0 \text{ and smooth } \cancel{\text{at } t=0}$$

where $J_p : E \to T_pE$ dash-to-dot map

$\xi \mapsto \dot{\gamma}(0)$.

where $\gamma(t) = p + t\xi$ path. $\xi = \gamma'(0)$

$\frac{d}{dt}\phi_t^* \tau \overset{\text{was proved before}}{=} \phi_t^*(L_{X_t}\tau) \overset{\text{Cartan}}{=} \phi_t^*(\iota(X_t)\cancel{d\tau} + d\iota(X_t)\tau) = d\sigma_t$. ↓ since w_i closed.

By def^n of σ_t, since τ vanishes on Q

$\Rightarrow \sigma_t$ vanishes on Q.

$$\tau = \tau - 0 = \phi_1^*\tau - \phi_0^*\tau$$
$$= \int_0^1 \frac{d}{dt}(\phi_t^*\tau)\, dt$$
$$= \int_0^1 (d\sigma_t)\, dt$$
$$= d\int_0^1 \sigma_t\, dt$$
$$= d\sigma. \quad \text{where } \sigma := \int_0^1 \sigma_t\, dt.$$

Hence. $\sigma|_{T_QM} = 0$ $\quad d\sigma = w_1 - w_0$ ✓.

Thm 3.2.2 (Darboux)

Every symplectic form w on M is locally diffeomorphic to the standard form w_0 on $\mathbb{R}^{2n}$

Pf: Apply 3.2.1 to the case where Q is a fixed point

$\Rightarrow$ ~~Every~~ Different symplectic forms on a nbhd of a fixed point are diffeo

Use 2.1.3: ~~Each symplectic form w is locally diffeo to w_0 on $\mathbb{R}^{2n}$~~ there is a vector space isomorphism between symplectic vector spaces (T_pM, w_p) and $(\mathbb{R}^{2n}, w_0)$

$\Rightarrow$ Every symplectic form w on M is locally diffeo to w_0 on $\mathbb{R}^{2n}$ (as a manifold)

[The Geometry of the Group of Symplectic Diffeomorphism]

Def. M: smooth manifold without boundary

$supp(\phi)$. $\quad Diff^c(M) = Diff(M) \cap \{$ with compact supports $\}$

Def. A <u>path</u> of diffeos is a map

$$f: I \to Diff^c(M)$$
$$t \mapsto f_t \quad \text{s.t.}$$

① $M \times I \to M$ $\quad (x,t) \mapsto f_t(x)$ is smooth.

② $\exists$ a compact subset K of M which contains $supp f_t$. $\forall t \in I$.

Denote this by $\{f_t\}$. If M is compact, ② is trivially satisfied.

Def: Hamiltonian vector field of F:
$i_\xi \Omega = -dF$. (different from another book)
$\mathrm{sgrad} F := \xi$. called skew gradient of F
ξ always exists and unique. ($\Leftarrow$ non-degenerate)

Def ~~F: M×I~~ Define $A := A(M)$
If M is closed. define $A(M)$ as the space of all smooth functions on M with zero mean with respect to the canonical volume form
If M is open, define $A(M)$ as ~~at~~ the space of all smooth functions with compact support

Def $I \subset \mathbb{R}$ be interval.
A (time-dependent) Hamiltonian function $F: M \times I \to \mathbb{R}$ is called normalized if $F_t \in A$, $\forall t$. when M is closed. When M is open, in addition we require $\exists$ a compact subset of M which contains the supports of all the functions F_t, $\forall t \in I$.

$\boxed{1.4}$

Def: $F: M \times I$ normalized Hamiltonian function.
Assume $0 \in I$.
$\{f_t\}$: flow of $\mathrm{sgrad} F_t$.
Say $\{f_t\}$ is the Hamiltonian flow generated by F
$\forall a \in I$ f_a is called a Hamiltonian diffeomorphism
Our definition implies that Hamiltonian diffeos are compactly supported. ✓ (see def^n for normalized F)

~~Def~~. $\mathrm{Ham}(M, \Omega) := \{$all H. diffeos$\}$.
A path of diffeos with values in $\mathrm{Ham}(M, \Omega)$ is called a Hamiltonian path.

Prop 1.4B: For every Hamiltonian path $\{f_t\}$, $t \in I$, there exists a unique (time-dependent) normalized Hamiltonian function $F : M \times I \to \mathbb{R}$ s.t.

$$\frac{d}{dt} f_t = \text{sgrad} F_t \circ f_t . \quad \forall t \in I.$$

The function F is called the <u>normalized Hamiltonian function of $\{f_t\}$</u>

<u>Pf</u> : First assume M satisfies : ~~Hdr~~ $H^1_{comp}(M;\mathbb{R}) = 0$

<u>examples</u> : 2-sphere : $\begin{cases} H^p(S^k) \cong \mathbb{R} & \text{for } p = 0, \text{or } k. \\ H^p(S^k) = 0 & \text{otherwise}. \end{cases}$ ✓.

linear space : Let M be contractible.

Then $\forall k \geq 1$. $H^k_{dR}(M) = 0$.

Denote by ξ_t the vector field generated by f_t

Since Hamiltonian diffeos preserve the symplectic form,

$0 = L_{\xi_t} \Omega \xrightarrow[d\Omega = 0]{\text{Cartan}} d i_{\xi_t} \Omega$. $\Rightarrow i_{\xi_t} \Omega$ is a closed form

$\Rightarrow i_{\xi_t} \Omega$ is exact.

$\Rightarrow \exists$ unique. (since normalized) smooth family of functions

$F_t(x) \in A$. s.t. $-dF_t = i_{\xi_t} \Omega$

$F(x,t)$ is normalized Hamiltonian of $\{f_t\}$.

If M s.t. $H^1_{comp}(M;\mathbb{R}) \neq 0$. it's complicated. ☒.

~~Rmk~~ . <u>Def</u> . $Symp(M,\Omega)$ = the group of all compactly supported diffeos f of M which preserves Ω.

Such diffeos are called <u>symplectomorphisms</u>

Denote by $Symp_0(M,\Omega)$:= path connected component of Id in $Symp(M,\Omega)$

~~If provided $H^1_{comp}(M;\mathbb{R}) = 0$.~~ $Ham(M,\Omega) \subseteq Symp(M,\Omega)$

~~we have:~~ is trivial ✓ (<u>since preserves Ω</u>)

If provided $H^1_{comp}(M;\mathbb{R}) = 0$. $Id \in Symp_0(M,\Omega)$, and

prove as <u>1.4B</u>. (since we only use $L_{\xi_t}\Omega = 0$ in <u>1.4B</u>)

, we get a normalized Hamiltonian function ~~F~~.

Hence, ~~Ham(M,Ω)~~ $Symp_0(M,\Omega) \subseteq Ham(M,\Omega)$

Hence. ~~Ham~~ $Symp_0(M,\Omega) = Ham(M,\Omega)$.

If $H^1_{comp}(M;\mathbb{R}) \neq 0$. there is not ~~static~~ true. ☒

(complicated)

prop 1.4D (Hamiltonian of the product)
Consider two Hamiltonian paths $\{f_t\}$ and $\{g_t\}$.
Let F, G be their normalized Hamiltonian functions.
Then the product path $h_t = f_t \circ g_t$ is a Hamiltonian path generated by the normalized Hamiltonian function $H(x,t) = F(x,t) + G(f_t^{-1}(x), t)$

Pf: $\frac{d}{dt}(f_t \circ g_t)(x)$

$$= \frac{\partial}{\partial t} f(g(x,t),t)$$
$$= \underbrace{\frac{\partial f}{\partial g}(g(x,t),t)\frac{\partial g}{\partial t}(x,t)}_{①} + \underbrace{\frac{\partial f}{\partial t}(g(x,t),t)}_{②}$$

$$② = \lim_{s\to 0} \frac{f_{t+s}(g(x,t)) - f_t(g(x,t))}{s}$$
$$= \frac{df_t}{dt}(g(x,t))$$
$$= \operatorname{sgrad} F_t \circ f_t \circ g_t(x).$$

$$① = \frac{\partial f_t}{\partial g}(g(x,t))\frac{\partial g}{\partial t}(x,t)$$
$$= (df_t)_{g(x,t)} \frac{dg_t}{dt}(x)$$
$$= (df_t)_{g(x,t)} \operatorname{sgrad} G_t \circ g_t(x)$$

consider $(f_t)_* (\operatorname{sgrad} G_t) \circ f_t \circ g_t(x)$
$$= (df_t)(f_t^{-1} \circ f_t \circ g_t(x)) \operatorname{sgrad} G_t (f_t^{-1} \circ f_t \circ g_t(x))$$
$$= (df_t)_{g(x,t)} \operatorname{sgrad} G_t \circ g_t(x)$$

So. $= (f_t)_* \operatorname{sgrad} G_t \circ f_t \circ g_t (x)$

Hence. ~~$\frac{d}{dt}(f_t \circ g_t) = \operatorname{sgrad} H_t \circ (f_t \circ g_t)$~~

~~where $H(x,t) =$~~

$$\frac{d}{dt}(f_t \circ g_t)(x) = \underbrace{(\operatorname{sgrad} F_t + (f_t)_* \operatorname{sgrad} G_t)}_{\cancel{H}} \circ f_t \circ g_t(x)$$

$$(f_t)_* \operatorname{sgrad} G_t = \operatorname{sgrad}(G_t \circ f_t^{-1})$$

Hence. $H(x,t) = F(x,t) + G(f_t^{-1}(x), t)$ is the Hamiltonian

and $\frac{d}{dt}(f_t \circ g_t) = \operatorname{sgrad} H_t \circ (f_t \circ g_t)$

$f_t \circ g_t \in \mathrm{Ham}(M, \Omega)$. ∎

prop 1.4E The set of Hamiltonian diffeos is a group with respect to composition

Pf: Take two Hamiltonian diffeos f and g.
write $f = f_1$, $g = g_1$ for some Hamiltonian flows $\{f_t\}$, $\{g_t\}$.
By 1.4D. ~~fg~~ $\{f_t g_t\}$ is also a Hamiltonian flow.
$\Rightarrow fg = f_1 g_1$ is also Hamiltonian diffeo.
For the inverse. $\{f_t^{-1}\}$:
need to find G_t s.t. $\frac{d}{dt} f_t^{-1} = \operatorname{sgrad} G_t \circ f_t^{-1}$ (Δ)
$f_t \circ f_t^{-1} = 1$.
$0 = \frac{d}{dt}(f_t \circ f_t^{-1}) \overset{1.4D}{=\!=} \operatorname{sgrad}(F_t + G_t \circ f_t^{-1}) \circ (f_t \circ f_t^{-1})$
$\Rightarrow$ $G(x,t) = -F(f_t x, t)$ satisfies (Δ)
$\Rightarrow$ $\{f_t^{-1}\}$ is also Hamiltonian flow.
Then use the same method as before. ▨

Def: Consider $\operatorname{Ham}(M,\Omega)$ as a Lie subgroup of $\operatorname{Diff}(M)$.
the Lie algebra of $\operatorname{Ham}(M,\Omega)$ = consists of ξ s.t. $\xi(x) = \frac{d}{dt}\big|_{t=0} f_t(x)$, since $\mathfrak{g} = T_e G$.
where $\{f_t\}$ is a smooth path on $\operatorname{Ham}(M,\Omega)$ with $f_0 = 1$
. Lie algebra of $\operatorname{Ham}(M,\Omega)$ can be identified with A;
Pf: ① $\forall \xi$, $\xi = \operatorname{sgrad} F_0$, since $\frac{d}{dt} f_t(x) = \operatorname{sgrad} F_t \circ f_t(x)$.
where F_t is the unique normalized Hamiltonian function generating the path. $F_0 = F(x,0) \in A$. ✓
② $\forall F \in A$. ~~sgrad F~~ take a normalized function s.t. $F_0 = F$. ~~take a Hamiltonian path~~
then $\operatorname{sgrad} F$ is the derivative at $t=0$ of the corresponding Hamiltonian flow. ▨

Def. Pick $f \in \operatorname{Ham}(M,\Omega)$, $G \in A$.
Let $\{g_t\}$, $g_0 = 1$ be a path on the group which is tangent to G, (i.e. $G = G_0$, where $G(x,t)$ is the normalized Hamiltonian ~~function~~)

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$$\boxed{Ad_f G = \frac{d}{dt}\Big|_{t=0} f g_t f^{-1}.}$$

since $Ad: G \to GL(\mathfrak{g})$
$g \mapsto D(g|e)$.
$c:\ c_f(g_t) = f g_t f^{-1}$

Differentiating:

$\frac{d}{dt} f g_t f^{-1}(x)$

$= \frac{d}{dt} f g_t(f^{-1}(x))$

$= \frac{\partial}{\partial t}(f(g(f^{-1}(x), t)))$

$= \frac{\partial f}{\partial g}(g(f^{-1}(x), t))\frac{\partial g}{\partial t}(f^{-1}(x), t)$

$= (df)_{g(f^{-1}(x), t)}\ \cancel{\text{sgrad } G_t}\ \frac{dg_t}{dt}(f^{-1}(x))$

$= (df)_{g(f^{-1}(x,t))}\ \text{sgrad } G_t \circ g_t \circ f^{-1}(x)$

$= f_*\ \text{sgrad} G_t\ (f \circ g_t \circ f^{-1}(x))$

$\Rightarrow Ad_f G = \frac{d}{dt}\Big|_{t=0} f g_t f^{-1} = \underbrace{f_*\ \text{sgrad} G\ (x)}_{\text{sgrad}(G \circ f^{-1})}.$

so we identify: $\underline{Ad_f G = G \circ f^{-1}}$

as the adjoint action of $Ham(M, \Omega)$ on A

Pick $F, G \in A$. with $\{f_t\}$. $f_0 = 1$. $F = F_0$.

Poisson bracket is defined on A as:

$$\{F, G\} := \frac{d}{dt}\Big|_{t=0} Ad_{f_t} G \overset{\text{By defn}}{=} \frac{d}{dt}\Big|_{t=0} G \circ f_t^{-1}$$

Rewrite $h_t := f_t^{-1}$

$\frac{d}{dt} G \circ f_t^{-1}(x)$

$= \frac{d}{dt} G(f_t^{-1}(x))$

$= \frac{d}{dt} G(h_t(x))$

$= dG\ (\text{sgrad } H_t)\ (G \circ f_t^{-1}(x))$

$H(x, t) = -F(f(x, t), t)$

$H_t = -F_t \circ f_t$

$\text{sgrad} H_t = \text{sgrad}(-F_t \circ f_t)$

$= -\text{sgrad}(F_t \circ (f_t^{-1})^{-1})$

$= -(f_t^{-1})_*\ \text{sgrad} F_t$

$= -(f_t)^*\ \text{sgrad } F_t$

At $t=0$. $\text{sgrad } H_0 = -\text{sgrad} F_0$

$\Rightarrow \{F, G\} = \frac{d}{dt}\Big|_{t=0} Ad_{f_t} G = -dG(\text{sgrad} F) \overset{\text{By defn}}{=} \Omega(\text{sgrad} F, \text{sgrad} G)$

~~Define~~ $L[X,Y] = L_X L_Y - L_Y L_X$.

~~Define $L_X L_Y$~~ We have. $[\mathrm{sgrad}\,F, \mathrm{sgrad}\,G] = -\mathrm{sgrad}\{F,G\}$

$\boxed{1.5}$.

<u>prop 1.5A</u> (M,Ω) closed symplectic manifold.

Then the group $\mathrm{Ham}(M,\Omega)$ is simple.

i.e. every normal subgroup is trivial, $\{1\}$ or itself

<u>prop 1.5C</u> Let $\{f_t\}$, $\{g_t\}$ be Hamiltonian flows generated by time-dependent normalized Hamiltonian functions F and G respectively. If $\forall t$ $f_t \circ g_t = g_t \circ f_t$

then $\{F,G\} = 0$

<u>Pf</u>. Use <u>1.4D</u>: Hamiltonians are equal

$$\hat F(x,t) + \hat G(f_t^{-1}x, t) = \hat G(x,t) + \hat F(g_t^{-1}x, t)$$

fix $t=0$. $F := \hat F_0$ $G := \hat G_0$

$$\Rightarrow F(x) + G(f_t^{-1}x) = G(x) + F(g_t^{-1}(x))$$

Differentiating at $t=0$.

$$\frac{d}{dt}\Big|_{t=0} G\circ f_t^{-1} = \frac{d}{dt}\Big|_{t=0} F\circ g_t^{-1}$$

i.e. $dG(-\mathrm{sgrad}\,F) = dF(-\mathrm{sgrad}\,G)$

$$\{F,G\} = \{G,F\}$$

anti symmetric. $\Rightarrow \{F,G\} = 0$ ∎

<u>prop 1.5B</u>. (M,Ω) sympl. mfd. $U\subset M$ non-empty open.

Then, there exists $f,g \in \mathrm{Ham}(M,\Omega)$ s.t. $\mathrm{supp}(f), \mathrm{supp}(g) \subset U$

and $fg \neq gf$.

<u>Pf</u>. Choose a point $x\in U$. $\xi, \eta \in T_xU$ s.t. $\Omega(\xi,\eta) \neq 0$

Choose germs of F, G s.t. $\mathrm{sgrad}\,F(x) = \xi$.

$\mathrm{sgrad}\,G(x) = \eta$.

Extend F, G by 0 outside U.

If M is open. By defn of normalized. don't need to add a const.

If M is closed. add a constant to ensure ~~normalized~~.

~~the~~ F, G have zero mean.

$\Rightarrow F, G \in \mathcal{A}$. constant outside U. $\Rightarrow$ f_t, g_t supported in U.

Since $0 \neq \Omega(\mathrm{sgrad}\,G, \mathrm{sgrad}\,F) = \{F,G\} \xRightarrow{1.5C} f_t, g_t$ not commutative ∎

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Th 1.5 D (M, Ω_1), (M, Ω_2) closed sympl. mfds

s.t. $Ham(M, \Omega_1) \cong Ham(M, \Omega_2)$

Then. they are conformally symplectomorphic:

i.e. $\exists$ a diffeo $f: M_1 \to M_2$.

$c \neq 0$.

s.t. $f^* \Omega_2 = c\,\Omega_1$.