I) Introduction + Recap:

Let G be a finite dim Lie Group w/ Lie Algebra $\mathfrak{g}$.
Let G be a finite dim Lie Group w/ Lie Algebra $\mathfrak{g}$.

A norm on $\mathfrak{g}$ is called invariant if it is preserved unde the adjoint action of G:

$$|\xi| = |g \xi g^{-1}| \quad \forall \xi \in \mathfrak{g},\ g \in G.$$

Any such action gives rise to a Metric on G:

$$d(g_0, g_1) := \inf_{g} \int_0^1 |\dot{g}\, g^{-1}|\, dt.$$

Where $g : g_0 \mapsto g_1$ is a path parameteized by t.

→ We want to ~~do~~ take this idea, and apply it to our setting:

As seen last time:

Given a symplectic Manifold (M, ω) and a smooth time dependent (normalized) Hamiltonian $H : M \times I \to \mathbb{R}$ we consider the vector field $X_{H_t} : M \times I \to TM$ given by $\omega(X_{H_t}, -) = dH_t$

~~We denote by Ham(X, ω) the~~

We then observe that $Symp_0(M, \omega)$, the path connected component of all diffeomorphisms of M, which preserve ω, containing the identity.

This turns out to be the same as $Ham(M, \omega)$ which is the Lie group ~~of flows ϕ_t of X_{H_t}~~ of flows $\phi_{t=a}$ for H_t norm..

We now ask the question: What is the minimal amount of Energy needed to generate a given Hamiltonian flow $\phi \in Ham(M,\omega)$

Let us formalize this Question:

Consider all possible Hamiltonian ~~flows~~ Paths $\{f_t\}$ $t \in [0,1]$ s.t. $f_0 = id$, $f_1 = \phi$. For each ~~flow~~ Path take it's unique normalized Hamiltonian H_t, and measure it's magnitude. Then minimize over all such ~~flows~~ Paths $\{f_t\}$.

Recall that H_t is an element of the Lie Algebra $\mathcal{A}$, (of $Ham(M,\omega)$)

Choose any norm on $\mathcal{A}$, $|\cdot|$ in a coordinate free way that is

$$\| H \circ \psi^{-1} \| = \| H \| \quad \forall H \in \mathcal{A},\ \psi \in Ham(M,\omega)$$

We now define the Magnitude as $\inf\limits_{H_t} \int_0^1 |H_t| \, dt$.

Observe this resemples some measurment of Length of $\{f_t\}$

This leads us to define

$$\ell(f_t) = \int_0^1 |H_t| \, dt.$$

Whish in turn leads natural to the definition of the pseudo metric on $Ham(M,\omega)$

$$\rho(\psi, \phi) = \inf_{f_t : \psi \to \phi} \int_0^1 |H_t| \, dt.$$

Lemma 1 Let F_t, G_t be normalized time dependent Hamiltonians generating paths ϕ_t and ψ_t. Then:

i) The path $\phi_t \psi_t$ is generated by $F_t + G_t \circ \phi_t^{-1}$

ii) The path ϕ_t^{-1} is generated by $- F_t \circ \phi$

iii) For $\psi \in Ham(M, \omega)$ $\psi^{-1} \phi_t \psi$ is generated by $F_t \circ \psi$

Proof: i) Let H_t be the Hamiltonian generating $\phi_t \psi_t$.

It must suffice $\partial_t \phi_t \psi_t = X_{H_t} \phi_t \psi_t$. $\qquad \omega(X_{H_t}, -) = dH_t$

Now $\partial_t \phi_t \psi_t = (\partial_t \phi_t) \psi_t + \phi_{t*} \partial_t \psi_t$

$$= X_{F_t} \phi_t \psi_t + \phi_{t*} X_{G_t} \psi_t$$

$$= X_{F_t} \phi_t \psi_t + X_{G_t \circ \phi_t^{-1}} \phi_t \psi_t$$

$$= (X_{F_t} + X_{G_t \circ \phi_t^{-1}}) \phi_t \psi_t.$$

ii) Consid $\partial_t \phi_t \phi_t^{-1} = 0$

$$X_{F_t} = - \phi_{t*} \partial_t \phi_t^{-1} \qquad (\Rightarrow \phi_t^* X_{F_t} = \partial_t \phi_t^{-1}$$

$$- X_{F_t} \circ \phi_t^{*} = \partial_t \phi_t^{-1}$$

iii) Exercice.

Theorem 1 ρ is a pseudometric.

i) $\rho(\phi_0, \phi_1) = \rho(\phi_1, \phi_0)$

ii) $\rho(\phi_0, \phi_2) \leq \rho(\phi_0, \phi_1) + \rho(\phi_1, \phi_2)$

iii) $\rho(\phi_0 \theta, \phi_1 \theta) = \rho(\theta \phi_0, \theta \phi_1) = \rho(\phi_0, \phi_1)$

iv) $\rho(\theta \phi_0 \psi^{-1}, \theta \phi_1 \theta^{-1}) = \rho(\phi_0, \phi_1)$

Proof: iii) Let H_t generate ϕ_t then $\theta\phi_0$ is generated by $H_t \circ \phi_0^{-1}$ and $\phi_0 \theta$ is generated by H_t. So by invariance ρ stays unchanged.

Observe due to iii) $\rho(\psi, \phi) = \rho(id, \phi\psi^{-1})$

ii) Choose compactly supported $F_t, G_t : M \times [0,1] \to \mathbb{R}$ generating ~~...~~ $\psi_t : id \mapsto \phi_1 \phi_0^{-1}$, $\eta_t : id \mapsto \phi_2 \phi_1^{-1}$.

Then $\xi_t : id \mapsto \phi_2 \phi_0^{-1}$ is generated by $K_t = G_t + F_t \circ \eta_t^{-1}$

Since $\|H_t\| \leq \|F_t\| + \|G_t\|$, the triangle inequality follows by taking inf over all F, G.

i) $L_t = -F_t \circ \psi_t$ generate $\phi_0 \phi_1^{-1}$ and clearly $|L_t| = |F_t|$.

iv) Lastly $F_t \circ \theta$ generates $\theta^{-1} \phi_1 \phi_0^{-1} \theta$ and clearly $|F_t| = |F_t \circ \theta|$. ∎

III) The question of non-degeneracy & the choise of Norm.

A very natural choise of Norms on $\mathcal{A}$ satisfying the invariancy assumption are the L_p norms.

$$\|H\|_p = \left(\int_M |H|^p \, dVol \right)^{1/p}$$

$$\|H\|_\infty = \max H - \min H$$

Theorem 1: The pseudodistance ρ_p for $p < \infty$ is degenerate. Moreove for closed Manifolds it vanishes identically

Theorem 3: For $p = \infty$ ρ_∞ is non degenerate in this case we call it the Hofer metric.

Our goal will be to proof Theorem 3 for the case $M = \mathbb{R}^{2n}$.

To that aim conside:

Definition: Let ρ be a biinvariant pseudo-metric on $Ham(M, \omega)$ and let A be a boundet subset of M.
The displacement enegy of A is given by:

$$e(A) := \inf \{ \rho(1, f) \mid f \in Ham(M, \omega) \; f(A) \cap A = \emptyset \}$$

with the convention $\inf \emptyset = +\infty$.

~~Clearly we have~~

Lemma 2: For every $A \subset B \subset M$ it holds:

i) $e(A) \leq e(B)$

ii) $e(f(A)) = e(A) \quad \forall f \in Ham(M, \omega)$

... $\blacksquare$

Proof: Exercice

Example: Conside $(\mathbb{R}^2, \omega)$ and take an open square A whose edges have length a & are parallel to the coordinate axes.

The corresponding system is:
Conside $H(p,q) = ap$, the corresponding system is:

$$\dot{q} = \frac{\partial H}{\partial p} = a$$

$$\dot{p} = -\frac{\partial H}{\partial q} = 0$$

~~Note all motion of~~

Therefore it's time 1 map h sends $(p,q) \mapsto (p, q+a)$

Note all motion of A takes place in $K = \text{Closure}(A \cup h(A))$.

Note all motion of A takes place in K = Closure. Conside $\hat{H}$ a cutoff of H outside a small nbhd of K.

Note $\hat{H}$ is a normalized Hamiltonian. And since $H = \hat{H}$ on K the time one map f generated by the Hamiltonian flow of $\hat{H}$

Observe $e(A) = \inf\{\rho(1,f) \mid f \in Ham(M,\omega)\ f(A) \cap A \neq \emptyset\} \leq \int_0^1 \max_K H - \min_K H \leq a^2$

So $e(A) \leq \text{Area}(A)$. As in $\mathbb{R}^2$ A is symplectomorphic to the disc we showed $e(B^2(r)) \leq \pi r^2$

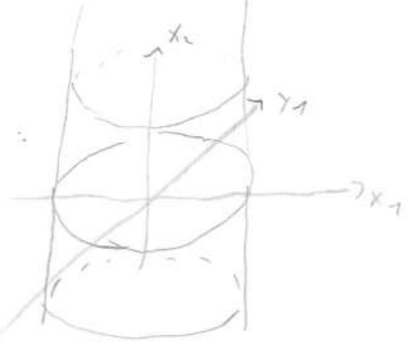
Theorem 4. The displacement energy is a relative symplectic capacity for subsets of $\mathbb{R}^{2n}$ and satisfies

$$e(B^{2n}(r)) = \pi r^2$$

Auxillary Theorem:

Theorem 5: (Non-squeezing Theorem) [Gromov]:
If there exists a symplectic embedding of $(B^{2n}(r), \omega_0)$ into $(Z^{2n}(R), \omega_0)$ then $r \leq R$.

Where $Z^{2n}(R) = \{(x,y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 \leq R\}$.



Picture: Consider $\mathbb{R}^4$ w/ coordinates (x_1, x_2, y_1, y_2) and the projection π onto the first 3 coordinates:
The $\pi(Z^{4}(R))$, and $\pi(B^4(r))$ look like:
Clearly there is a linear map taking $B^{2n}(r) \hookrightarrow Z^{2n}(R)$ for $r > R$.

$$\begin{pmatrix} \frac{R}{r}a & & & \\ & a & & \\ & & \frac{R}{r}a & \\ & & & a \end{pmatrix} \qquad \sqrt[2n-2]{\left(\frac{r}{R}\right)^2} = a \quad \left(\frac{r}{R}\right)^{\frac{2}{2n-2}}$$

Def: The Gromov width of a symplectic Manifold (M, ω)

$$w_G(M) := \sup\{r^2\pi \mid B^{2n}(r) \text{ embedds symplectically into } M\}.$$

It holds:

i) $M_1\omega_1 \hookrightarrow M_2\omega_2$ and $\dim M_1 = \dim M_2 \Rightarrow w_G(M_1) \leq w_G(M_2)$

ii) $w_G(M, \lambda\omega) = \lambda\, w_G(M, \omega)$

iii) $w_G(B^{2n}(r)) > 0$
$w_G(Z^{2n}(r)) < \infty$.

Proof of Theorem 4: $\left(e(B) = \pi r^2 \right)$ (Sketch)

$e(B) \leq \pi r^2$ ✓

$e(B) \geq \pi r^2$:

Assume ϕ_1 is a compactly supported Hamiltonian diffeo satisfying $\phi(B) \cap B = \emptyset$. Denote by

$$c := w_G(B) = \pi r^2 , \quad e = \rho(\phi_1, id)$$

If we can show $e \geq c$ we are done.

To this aim let us construct $\phi := \psi \circ \phi_1 \circ \psi^{-1}$ such that $\phi(B) \cap \sqrt{2}\, B = \emptyset$. This symplectomorphism has the same dist to the identity as ϕ_1.

Using ϕ we shall construct for every $\varepsilon > 0$ an embedding

$$B^{2n+2}(\sqrt{2}\, r) \hookrightarrow Z^{2n+2}(R) = B^2(R) \times \mathbb{R}^{2n} \text{ s.t.}$$

Observe that $\omega_G(B^{2n+2}(\sqrt{2}\, r)) = 2c$ ~~and~~ $\underline{\text{s.t. } \omega_G(Z^{2n+2n}(R)) = \pi R^2 = e + e + \varepsilon}$

Now by Gromov's non squeezing Theorem we must have

$$2c \leq e + e + \varepsilon$$
$$c \leq e + \varepsilon$$

Proof of Theorem 3:

Let ϕ be a symplectomorphism of $(\mathbb{R}^{2n}, \omega_0)$ not equal to the identity. Then there exists a Ball of radius r s.t. $\phi(B) \cap B = \emptyset$.

Hence $e(B) \leq \rho(id, \phi)$. $\blacksquare$