Lagrangian submanifolds

Goal: prove that Hofer's metric on $\mathbb{R}^{2n}$ is non-degenerate
→ we introduce Lagrangian submanifolds.

Definition

Let (M^{2n}, Ω) a symplectic manifold and let $L \subset M$ a submanifold.
We call L **Lagrangian** if
- $\dim L = \frac{1}{2} \dim M = n$
- $\Omega|_{TL} \equiv 0$

An embedding (or immersion) $\varphi : L^n \to M^{2n}$ is called Lagrangian if $\varphi^* \Omega \equiv 0$.

Example:

1) Curves on surfaces:
Let (M^2, Ω) be an oriented surface with an area form.
Claim: Every curve is Lagrangian
Proof: · The tangent space to a curve is one-dimensional
· Ω vanishes when it's evaluated on two proportional vectors.

2) Let N^n be a any manifold.
Consider the cotangent bundle $M = T^*N$ of N with the natural projection $\pi : T^*N \to N$, $(p,q) \mapsto q$.
We define the following 1-form λ on M which is called the Liouville form.
For $q \in N$, $(p,q) \in T^*N$ and $\xi \in T_{(p,q)}T^*N$ we set $\lambda(\xi) = \langle p, \pi_* \xi \rangle$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between T^*N and TN.

Claim: $\Omega = d\lambda$ is a symplectic form on T^*M.
Proof: We use local coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ on T^*N and we write $\xi = (\dot{p}_1, \ldots, \dot{p}_n, \dot{q}_1, \ldots, \dot{q}_n)$.
So $\pi_* \xi = (\dot{q}_1, \ldots, \dot{q}_n)$.
With this notation we have $\langle p, \pi_* \xi \rangle = \sum p_i \dot{q}_i$ and then follows $\lambda(\xi) = \sum p_i \, dq_i$.
Therefore $\Omega = d\lambda = \sum dp_i \wedge dq_i$, which is the standard symplectic form on $\mathbb{R}^{2n}$. $\square$

3) Lagrangian suspension:
Let $L \subset (M, \Omega)$ be a Lagrangian submanifold.
Consider a loop of Hamiltonian diffeomorphism $\{h_t\}$, $t \in S^1$, $h_0 = h_1 = \mathbb{1}$, generated by a 1-periodic Hamiltonian function $H(x,t)$.
These are going to be usefull later.

Proposition:

Let $M \times T^*S^1$ be a symplectic manifold with symplectic form $\sigma = \Omega + dr \wedge dt$.
M as before and (r,t) are the coordinates on $T^*S^1 = \mathbb{R} \times S^1$.
Then
$$\phi : L \times S^1 \longrightarrow M \times T^*S^1$$
$$(x,t) \longmapsto (h_t(x), -H(h_t(x),t), t)$$
is a Lagrangian embedding.

Proof:

Enough to show: $\phi^*\sigma$ vanishes on pairs (ξ,ξ') and on pairs $(\xi, \frac{\partial}{\partial t})$ for $\xi, \xi' \in TL$ and $\frac{\partial}{\partial t} \in TS^1$.
We only need these to show these two pairs because the other possible pairs vanishes by antisymmetry.

By computing, we obtain: $\phi_* \xi = \frac{\partial}{\partial s} \phi(\gamma(s),t)|_{s=0}$ for $\gamma : (-\varepsilon, \varepsilon) \to L$, $\dot\gamma(0) = \xi$ and $\gamma(s) = x$.

$$= \frac{\partial}{\partial s} (h_t(\gamma(s)), -H_t(h_t(\gamma(s)),t)|_{s=0}$$
$$= (dh_t)_{\gamma(0)}(\xi) - (dH_t)_{h_t(\gamma(0))}((dh_t)(\xi)) \frac{\partial}{\partial r}$$ using the pushforward
$$= h_{t*}\xi - \langle dH_t, h_{t*}\xi \rangle \frac{\partial}{\partial r}$$

and analogously $\phi_* \xi' = h_{t*}\xi' - \langle dH_t, h_{t*}\xi' \rangle \frac{\partial}{\partial r}$

$$\Longrightarrow \phi^*\sigma(\xi,\xi') = \sigma(\phi_*\xi, \phi_*\xi') = \Omega(h_{t*}\xi, h_{t*}\xi') = \Omega(\xi,\xi') = 0$$
└ L Lagrangian

- $\phi_* \frac{\partial}{\partial t} = \frac{d}{dt} (h_t(x), -H_t(h_t(x)), t)$

$$= \underbrace{\text{sgrad}\, H_t(h_t)}_{\text{skew gradient of } H_t} - \left(\frac{\partial H}{\partial t}(h_t(x)) - dH_t(\text{sgrad}\, H_t(h_t)) \right) \frac{\partial}{\partial r} + \frac{\partial}{\partial r}$$

$$= \text{sgrad}\, H_t - \left(\langle dH_t, \text{sgrad}\, H_t \rangle + \frac{\partial H}{\partial t} \right) \frac{\partial}{\partial r} + \frac{\partial}{\partial t}$$

$$= \text{sgrad}\, H_t - \frac{\partial H}{\partial t} \frac{\partial}{\partial r} + \frac{\partial}{\partial t}$$

$$\Longrightarrow \phi^*\sigma(\xi, \tfrac{\partial}{\partial t}) = \Omega(h_{t*}\xi, \text{sgrad}\, H_t) + dr \wedge dt \left(-\langle dH_t, h_{t*}\xi \rangle \frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right)$$
$$= \Omega(h_{t*}\xi, \text{sgrad}\, H_t) - \langle dH_t, h_{t*}\xi \rangle$$
$$= dH_t(h_{t*}\xi) - \langle dH_t, h_{t*}\xi \rangle$$
$$= 0$$

Then ϕ is a Lagrangian embedding. $\square$

The Liouville class of Lagrangian submanifolds in $\mathbb{R}^{2n}$:

Let $L \subset (\mathbb{R}^{2n}, dp \wedge dq)$ be a Lagrangian submanifold.

Let $\lambda = p_1 dq_1 + \_\_ + p_n dq_n$ the Liouville form, and $\lambda_{|T_L}$ a restriction to L.

Note: $d(\lambda_{|T_L}) = \Omega_{|T_L} = 0$

Lagrangian submanifold

Definition:

> The cohomology class $\lambda_L \in H^1(L, \mathbb{R})$ of this closed 1-form is called **Liouville class** of the Lagrangian submanifold L.
> Similarly, for a Lagrangian embedding or immersion $\phi : L \to \mathbb{R}^{2n}$, we define the **Liouville class** as $[\phi^* \lambda]$.

Geometrical interpretation: (of the Liouville class of a Lagrangian submanifold)

Let $H_1(\mathbb{R})$ be a 1-cycle.

We can then find a 2-chain Σ in $\mathbb{R}^{2n}$ with $\partial\Sigma = a$.

$(\lambda_L, a) = \int_a \lambda_L = \int_\Sigma \Omega$

Note: independent of Σ.

(λ_L, a) is called the **symplectic area** of a.

- λ_L is invariant under symplectomorphisms of $\mathbb{R}^{2n}$, i.e. $\varphi^* \lambda_{\varphi(L)} = \lambda_L$

Theorem A:

> Assume that $L \subset \mathbb{R}^{2n}$ is a closed Lagrangian submanifold.
> Then $\lambda_L \neq 0$.

Antonio will prove it next time

⚠ For a Lagrangian immersions this statement is not true in general.

Example case $n = 1$:

Any closed embedded curve bounds a subset of positive area.

But, if we take for example the immersed figure eight curve can bound a set with zero area.

Definition:

> A closed Lagrangian submanifold $L \subset (\mathbb{R}^{2n}, \omega)$ is called **rational** iff $\lambda_L(H_1(L; \mathbb{Z})) \subset \mathbb{R}$ is a discrete subgroup.
> We denote its positive generator by $\gamma(L)$.

Example: the split torus (which is Lagrangian) $L = S^1(r) \times \_\_ \times S^1(r) \subset \mathbb{R}^{2n}$ is rational

Proof: each $S^1(r)$ has a symplectic area πr^2, and then $\gamma(L) = \pi r^2$.

However, the torus $S^1(1) \times S^1(\sqrt[3]{2}) \subset \mathbb{R}^4$ is not rational, since the symplectic areas of the two circles are π and $\sqrt[3]{4}\,\pi$ and they generate a dense subgroup of $\mathbb{R}$.

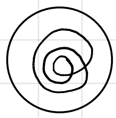
Theorem B:

> Let $L \subset B^2(r) \times \mathbb{R}^{2n-2}$ be a closed rational Lagrangian submanifold
> Then $\gamma(L) \leq \pi r^2$.

Antonio will prove it next time

⚠ The condition that L is embedded is necessary.

Ex: this is an immersed Lagrangian submanifold of arbitrary symplectic area:



Theorem C:

Let $L \subset \mathbb{R}^{2n}$ a closed rational Lagrangian submanifold.
Then $e(L) \geq \frac{1}{2}\gamma(L)$

Proof: follows from thm B, see below.

Rk: this result gives a lower bound for the displacement energy $e(L)$ of a rational Lagrangian submanifold L wrt Hofer's metric.

Consequences:

Non-degeneracy of Hofer's metric:

Thm. C implies that Hofer's metric on $Ham(\mathbb{R}^{2n}, \omega)$ is non-degenerate.

Proof: Let $B^{2n}(r) = \{ p_1^2 + \ldots + p_n^2 + q_1^2 + \ldots q_n^2 \leq r^2 \}$. Each ball $B^{2n}(r)$ contains a rational split torus $S^1(\frac{r}{\sqrt{n}}) \times \ldots \times S^1(\frac{r}{\sqrt{n}}) = \{ p_1^2 + q_1^2 = \frac{r^2}{n}, \ldots, p_n^2 + q_n^2 = \frac{r^2}{n} \}$.

Thus $e(B^{2n}(r)) \overset{\text{Thm C}}{\geq} \frac{\pi r^2}{2n} > 0 \overset{\text{last}}{\underset{\text{week}}{\Longrightarrow}}$ non-degeneracy of ρ.

[Actually Hofer proved $e(B^{2n}(r)) = \pi r^2$.]

Estimating the displacement energy:

Goal of this section: prove thm C using thm B.

Let ρ is a biinvariant pseudo-distance on $Ham(M,\Omega)$ and let A a bounded subset of M. | as already seen

Definition

The displacement energy of A is given by

$$e(A) = \inf\{\rho(1\!\!1,\varphi) \mid \varphi \in Ham(M,\Omega), \underbrace{\varphi(A) \cap A = \phi}_{\text{"A get displaced by } \varphi\text{"}}\}$$

- The set of φ may be empty
- convention: $\inf \phi = \infty$
- if $e(A) \neq 0$ we say that A has positive displacement energy.

Properties (skip):
- monotone: $A \subset B \Rightarrow e(A) \leq e(B)$
- invariant: $e(A) = e(\varphi(A))$ for every Hamiltonian diffeomorphism φ of M. (i.e. $\forall \varphi \in Ham(M,\omega)$)

To prove thm C, we proceed in 5 steps:

Step 1: Let L be a closed rational Lagrangian submanifold. Let $h_t, t \in [0,1]$ be a path of Hamiltonian diffeomorphism s.t. $h_0 = 1\!\!1$ and $h_1(L) \cap L = \phi$.

Fix $\varepsilon > 0$.

wlog we can assume that $h_t = 1\!\!1$ for $t \in [0,\varepsilon]$ and $h_t = h_1$ for $t \in [1-\varepsilon, 1]$

Indeed, this can be achieved by suitable reparam. of the flow which preserves the length:

Ex: Let $\{\varphi_t\}$, $t \in [0,a]$ be a Hamiltonian flow generated by a normalized Hamiltonian $F(x,t)$.
Then $\{\varphi_{at}\}$, $t \in [0,1]$ is again a Hamiltonian flow generated by $aF(x,at)$.
In fact, every Hamiltonian diffeomorphism is a time-one map of some Hamiltonian.

Let $H(x,t)$ the corresponding Hamiltonian function.
Set $\ell = \text{length}\{h_t\} = \int_0^1 \max_x H_t - \min_x H_t \, dt$

t.s: $\ell \geq \frac{1}{2}\gamma(L)$ (thm C.)

Step 2: We create a loop of Hamiltonian diffeomorphisms for $t \in [0,2]$.

$$g_t = \begin{cases} h_t & t \in [0,1] \\ h_{2-t} & t \in [1,2] \end{cases}$$

with Hamiltonian

$$G(x,t) = \begin{cases} H(x,t) & t \in [0,1] \\ -H(x,2-t) & t \in [1,2] \end{cases}$$

Claim: $\int_0^2 G(g_t(x),t)\,dt = 0 \quad \forall x$. (we use it later ☆)

Proof: $\int_0^2 G(g_t(x),t)\,dt = \int_0^1 G(g_t(x),t)\,dt + \int_1^2 G(g_t(x),t)\,dt$
(skip)

$$= \int_0^1 H(h_t(x),t)\,dt + \int_1^2 -H(h_{2-t},2-t)\,dt$$

$$= \int_0^1 H(h_t(x),t)\,dt - \underbrace{\int_1^2 H(h_{2-t},2-t)\,dt}_{+\int_0^1 H(h_{\tilde t},\tilde t)(\cancel{-}dt)}$$

$\tilde t = 2-t$, $d\tilde t = -dt$, $\int_1^2 \to \int_1^0 = -\int_0^1$

$$= 0$$

We now apply Ex 3 (Lagrangian suspension) and we get a new Lagrangian submanifold $L' \subset \mathbb{R}^{2n} \times T^*S^1$ as the image of $L \times S^1$ under the mapping

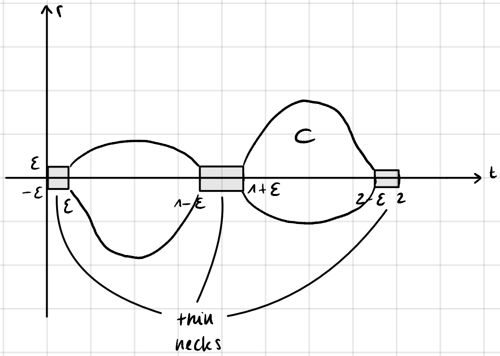
$$(x,t) \mapsto (g_t(x), -G(g_t(x),t), t)$$

Note: here $S^1 = \mathbb{R}/2\mathbb{Z}$.

We define two function: $a_+(t) = -\min_x G(x,t) + \varepsilon$ & $a_-(t) = -\max_x G(x,t) - \varepsilon$

$\Rightarrow L' \subset \mathbb{R}^{2n} \times C \subset \mathbb{R}^{2n} \times T^*S^1$

annulus $\{a_-(t) < r < a_+(t)\}$



Step 3:

Goal: pass from $\mathbb{R}^{2n} \times T^*S^1$ to $\mathbb{R}^{2n} \times \mathbb{R}^2$.

In order to do that we consider a particular symplectic immersion $\Theta : C \to \mathbb{R}^2$ (Gromov's figure eight trick)

$$\mathrm{area}(C) = \int_0^2 (a_+(t) - a_-(t))\,dt \overset{DEF}{=} \int_0^2 (-\min(G(x,t)) + \varepsilon + \max(G(x,t)) + \varepsilon)\,dt$$

$$= \int_0^2 (\max(G(x,t)) - \min(G(x,t)) + 2\varepsilon)\,dt = \int_0^2 (\max(G(x,t)) - \min(G(x,t))\,dt + 4\varepsilon$$

$$= \int_0^1 \max(H(x,t)) - \min(H(x,t))\,dt + \underbrace{\int_1^2 \max(-H(x,2-t)) - \min(-H(x,2-t))\,dt}_{=\int_0^1 \max(-H(x,\tilde t)) - \min(-H(x,\tilde t))\,d\tilde t} + 4\varepsilon$$

$\tilde t = 2-t$ $\int_1^2 \to \int_1^0 = -\int_0^1$ $dt = -d\tilde t$

$$= \int_0^1 -\min(H(x,\tilde t)) + \max(H(x,\tilde t))\,d\tilde t$$

$$= 2\int_0^1 (\max_x H_t - \min_x H_t)\,dt + 4\varepsilon$$

$$= 2\ell + 4\varepsilon$$

$\Rightarrow$ the image $\Theta(C)$ can be enclose by a disc B of area $2\ell + 10\varepsilon$

Step 4:

Consider the symplectic immersion $\Theta' = id \times \Theta : \mathbb{R}^{2n} \times C \to \mathbb{R}^{2n} \times \mathbb{R}^2$

- $\Theta'(L') \subset \mathbb{R}^{2n+2}$ is an immersed Lagrangian submanifold and $\Theta'(L') \subset \mathbb{R}^{2n} \times B$

Claim: $L'' = \Theta'(L')$ is embedded:

Proof: note: the only place where double points could occur is in the thin necks.
For $t \in [-\varepsilon, \varepsilon]$: $g_t(L) = h_0(L) \overset{\mathbb{1}}{=} L$
& for $t \in [1-\varepsilon, 1+\varepsilon]$: $g_t(L) = h_1(L)$, and by assumption on step 1 $h_1(L) \cap L = \emptyset$
$\Rightarrow$ NO double points and Θ' is an embedding. $\square$

Step 5:

t.s: L'' rational (so we can apply thm B)
We will prove $\gamma(L) = \gamma(L'')$.

Let ϕ be the composition of the Lagrangian suspension and Θ';

$$\phi : L \times S^1 \xrightarrow[\text{L. susp.}]{} \mathbb{R}^{2n} \times T^* S^1 \xrightarrow{\Theta'} \mathbb{R}^{2n}(p_1, \dots, p_n, q_1, \dots, q_n) \times \mathbb{R}^2(p, q)$$

$$(x, t) \longmapsto (g_t(x), \Theta(-G(g_t(x), t), t))$$

Note: L'' is the image of $L \times S^1$ under ϕ.
The group $H_1(L'')$ is generated by the cycles of the form $\phi(b)$ where b case 1: $\subset L \times \{0\}$; case 2: $= \{x_0\} \times S^1$, $x_0 \in L$

- case 1: $\phi(b) = b \times \{\Theta(0,0)\}$ $\Rightarrow$ symplectic area of b and $\phi(b)$ coincide
- case 2: Let α be the orbit $\{g_t x_0\}$, $t \in [0,2]$.

$$\text{Note: } \int_b \phi^*(p_1 dq_1 + \dots + p_n dq_n + p dq) = \int_\alpha p_1 dq_1 + \dots + p_n dq_n + \int \Theta^* p dq$$

$$= 0 + \int r dt = -\int_0^2 G(g_t(x_0), t) dt \overset{\star}{=} 0$$

$\Rightarrow L''$ is a rational Lagrangian submfd with $\gamma(L) = \gamma(L'')$.

Since $L'' \subset \mathbb{R}^{2n} \times B$, by thm B: $\gamma(L'') \underset{\text{thm B}}{\leq} \text{area}(B) = 2e + 10\varepsilon \quad \forall \varepsilon > 0$.

$\Rightarrow e(L) \geq \frac{1}{2}\gamma(L)$

Second book:

Theorem

Let (M,ω) a symplectic manifold and $L \subset M$ be a compact Lagrangian submanifold.
Then there exists a neighbourhood $\mathcal{N}(L_0) \subset T^*L$ of the zero section, a nbhd $V \subset M$ of L,
and a diffeomorphism $\phi : \mathcal{N}(L_0) \to V$ s.t. $\phi^*\omega = -d\lambda$

$$\phi_{|L} = id$$

where λ is the canonical 1-form on T^*L.

Proof:

The proof is based on the fact that the normal bundle of L in M is isomorphic to the tangent bundle.
To define such isomorphism, we use a compatible complex structure J on the tangent bundle TM (exists by a Prop.)

By an another Proposition we have that the subspace $J_q T_q L \subset T_q M$ is the orthogonal complement of $T_q L$ wrt the metric g_J induced by J, and is a Lagrangian subspace of $(T_q M, \omega)$.

Let $\Phi_q : T_q^* L \to T_q L$ the isomorphism induced by the metric g_J, i.e. $g_J(\Phi_q(v^*), v) := v^*(v)$, $v \in T_q L$

Now consider $\phi : T^*L \to M$

$$\phi(q, v^*) := \exp_q(J_q \Phi_q(v^*))$$

exponential map of the Riemannian metric g_J.

Then for $v = (v_0, v_1^*) \in T_q L \oplus T_q^* L = T_{(q,0)} T^*L$ we have $d\phi_{(q,0)}(v) = v_0 + J_q \Phi_q(v_1^*)$

$$\Rightarrow \phi^*\omega_{(q,0)}(v,w) = \omega_q\left(v_0 + J_q\Phi_q(v_1^*),\, w_0 + J_q \Phi_q(w_1^*)\right)$$

$$= \omega_q(v_0, J_q\Phi_q(w_1^*)) - \omega_q(w_0, J_q \Phi_q(v_1^*))$$

$$= g_J(v_0, \Phi_q(w_1^*)) - g_J(w_0, \Phi_q(v_1^*))$$

$$= w_1^*(v_0) - v_1^*(w_0)$$

$$= -d\lambda_{(q,0)}(v,w)$$

$\Rightarrow$ the 2-form $\phi^*\omega \in \Omega^2(T^*L)$ agrees with the canonical 1-form $-d\lambda_{can}$ on the zero section

Done by Moser isotopy.

Lemma (Moser Isotopy):

Let M be a $2n$-dim. smooth mfd and $Q \subset M$ be a compact submfd.
Suppose that $\omega_0, \omega_1 \in \Omega^2(M)$ are closed 2-forms s.t. at each point q of Q the forms ω_0 and ω_1 are equal and non-degenerate on $T_q M$. Then $\exists$ open nbhds $\mathcal{N}_0$ and $\mathcal{N}_1$ of Q and a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}_1$ s.t. $\psi_Q = id$

$$\psi^*\omega_1 = \omega_0$$