Pseudoholomorphic Curves

Antonio Casetta

Seminar: Introduction to Hofer's Geometry
Supervisor: Dr. J. Chassé
ETH Zürich - DMATH
FS 2023

The main references of this work is the books "The Geometry of the Group of Symplectic Diffeomorphisms" by L. Polterovich.

In this pages we prove the theorem which states that $\gamma(L) \leq \pi r^2$ for any closed rational Lagrangian submanifold $L \subset B^2(r) \times \mathbb{R}^{2n-2}$. The proof is based on Gromov's techniques of pseudo-holomorphic discs.

1 Introducing the $\bar{\partial}$-operator

Identify $\mathbb{R}^{2n}(p_1, q_1, \ldots, p_n, q_n)$ with

$$\mathbb{C}^n(p_1 + iq_1, \ldots, p_n + iq_n) = \mathbb{C}^n(w_1, \ldots, w_n).$$

Denote by $\langle\ ,\ \rangle$ the Euclidean scalar product. The three geometric structures we get in this way are the Euclidean, the symplectic and the complex structure. They are related by the following formula

$$\langle \xi, \eta \rangle = \omega(\xi, i\eta)$$

where $\omega = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q_n$. We will check this formula in the case $n = 1$. Let $\xi = (p', q')$ and $\eta = (p'', q'')$. Then

$$\mathrm{d}p \wedge \mathrm{d}q(\xi, i\eta) = \mathrm{d}p \wedge \mathrm{d}q \left(\begin{pmatrix} p' \\ q' \end{pmatrix}, \begin{pmatrix} -q'' \\ p'' \end{pmatrix} \right) = p'p'' + q'q'' = \langle \xi, \eta \rangle.$$

In what follows we will measure areas and lengths using the Euclidean metric. Consider the unit disc $D \subset \mathbb{C}$ with coordinate $z = x + iy$.

Definition 1.1. For a smooth map $f : D \to \mathbb{C}^n$ we define the **$\bar{\partial}$-operator**,

$$\bar{\partial} : C^\infty(D, \mathbb{C}^n) \to C^\infty(D, \mathbb{C}^n),$$
$$f \mapsto \bar{\partial} f = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

Example 1.1. Let $f : \mathbb{C} \to \mathbb{C}, z \mapsto \bar{z}$. So $f(x, y) = x - iy$ and $\bar{\partial} f = \frac{1}{2}(1 + 1) = 1$. We observe that $\bar{\partial} f = \frac{\partial f}{\partial \bar{z}}$.

Let us introduce two useful geometric quantities associated with a map $f : D \to \mathbb{C}^n$.

Definition 1.2. The **symplectic area** of $f : D \to \mathbb{C}^n$ is given by

$$\omega(f) = \int_D f^* \omega$$

and the **Euclidean area** of f is given by

$$\mathrm{Area}(f) = \int_D \sqrt{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle \left\langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y} \right\rangle - \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle^2}\, \mathrm{d}x\mathrm{d}y.$$

Proposition 1.1. (i) $\mathrm{Area}(f) \le 2\int_D |\bar\partial f|^2 \, \mathrm{d}x\mathrm{d}y + \omega(f)$,

(ii) $\mathrm{Area}(f) \ge |\omega(f)|$.

Proof. (i) Given $\xi, \eta \in \mathbb{C}^n$ we have the following inequality:

$$\sqrt{|\xi|^2|\eta|^2 - \langle \xi, \eta\rangle^2} \le |\xi||\eta| \le \frac{1}{2}\left(|\xi|^2 + |\eta|^2\right).$$

But

$$\frac{1}{2}|\xi + i\eta|^2 + \omega(\xi,\eta) = \frac{1}{2}\left(|\xi|^2 + |\eta|^2\right) + \langle \xi, i\eta\rangle + \langle \xi, -i\eta\rangle = \frac{1}{2}\left(|\xi|^2 + |\eta|^2\right),$$

so we get

$$\sqrt{|\xi|^2|\eta|^2 - \langle \xi, \eta\rangle^2} \le \frac{1}{2}|\xi + i\eta|^2 + \omega(\xi,\eta).$$

So, putting $\xi = \frac{\partial f}{\partial x}$ and $\eta = \frac{\partial f}{\partial y}$, we get

$$\mathrm{Area}(f) \le \int_D \frac{1}{2}\left|\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right|^2 + \omega\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \, \mathrm{d}x\mathrm{d}y = 2\int_D |\bar\partial f|^2 \, \mathrm{d}x\mathrm{d}y + \omega(f).$$

(ii) If $\eta \ne 0$ note that $\eta,\ i\eta$ are orthogonal i.e. $\langle \eta, i\eta\rangle = 0$. Projecting ξ on η and $i\eta$ we get

$$\left\langle \xi, \frac{\eta}{|\eta|}\right\rangle^2 + \left\langle \xi, \frac{i\eta}{|i\eta|}\right\rangle^2 \le |\xi|^2$$

Since $|\eta| = |i\eta|$ the above reads

$$\langle \xi, \eta\rangle^2 + \omega(\xi,\eta)^2 \le |\xi|^2|\eta|^2$$

hence

$$|\omega(\xi,\eta)| \le \sqrt{|\xi|^2|\eta|^2 - \langle \xi, \eta\rangle^2}.$$

So

$$\mathrm{Area}(f) \ge \int_D \left| \omega\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right| \, \mathrm{d}x\mathrm{d}y \ge \left|\int_D \omega\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \, \mathrm{d}x\mathrm{d}y\right| = |\omega(f)|.$$

□

2 The boundary value problem

Definition 2.1. Let (M^{2n}, Ω) be a symplectic manifold and let $L \subset M$ be a submanifold. We say that L is a **Lagrangian submanifold** if $\dim L = \frac{1}{2}\dim M = n$ and $\Omega|_{TL} \equiv 0$. An embedding (or immersion) $f : L^n \to M^{2n}$ is called Lagrangian if $f^*\Omega \equiv 0$.

Let $L \subset \mathbb{C}^n$ be a closed Lagrangian submanifold and let $g : D \times \mathbb{C}^n \to \mathbb{C}^n$ be a smooth map which is bounded together with all its derivatives. Fix a homology class $\alpha \in H_2(\mathbb{C}^n, L)$. Consider the following problem. Find a smooth map $f : (D, \partial D) \to (\mathbb{C}^n, L)$ such that

$$\begin{cases} \bar\partial f(z) = g(z, f(z)) \\ [f] = \alpha. \end{cases} \tag{$P(\alpha,g)$}$$

Example 2.1. If $g = 0,\ \alpha = 0$ then the space of solutions of $P(0,0)$ consists of the constant mappings $f(z) \equiv w$ for $w \in L$. To see this first observe that $\omega(f) = 0$. Indeed, since $\alpha = 0$ and L is Lagrangian, the curve $f(\partial D)$ bounds a 2-chain in L with zero symplectic area. This chain together with $f(D)$ forms a closed surface in $\mathbb{C}^n$. Since ω is exact, the symplectic area of this surface vanishes. Therefore $\omega(f) = 0$. Further, since $g = 0$ we get that $\bar{\partial} f = 0$. So the first part of previous proposition yields $\mathrm{Area}(f) = 0$ and hence $\frac{\partial f}{\partial x}$ is parallel to $\frac{\partial f}{\partial y}$. On the other hand $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ and therefore $\frac{\partial f}{\partial x} \perp \frac{\partial f}{\partial y}$. Consequently $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. So f is a constant map and because of the boundary condition its image lies in L.

Assume now that we have a sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ which C^∞-converges to some function g. Let f_n be the solutions of the corresponding problems $P(\alpha, g_n)$. Gromov's famous compactness theorem states that either

(i) $\{f_n\}_{n\in\mathbb{N}}$ contains a subsequence that converges to a solution of $P(\alpha, g)$ or

(ii) *bubbling off* takes place.

In order to explain what bubbling off is we introduce the concept of a cusp solution.

Definition 2.2. Consider the following data:

1. a decomposition $\alpha = \alpha' + \beta_1 + \cdots + \beta_k$, where $\forall j \in \{1, \ldots, k\} : \beta_j \neq 0$;
2. a solution f of $P(\alpha', g)$;
3. solutions h_j of $P(\beta_j, 0)$, $j \in \{1, \ldots, k\}$, called **pseudo-holomorphic discs**.

This object is called a **cusp solution** of $P(\alpha, g)$ and $f(D) \cup h_1(D) \cup \cdots \cup h_k(D)$ is called its image.

The bubbling off phenomenon means that there exists a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ which converges to a cusp solution of $P(\alpha, g)$. The only feature of this convergence which is important for our purposes is the continuity of the Euclidean area:

$$\mathrm{Area}(f_n) \longrightarrow \mathrm{Area}(f) + \sum_{j=1}^{k} \mathrm{Area}(h_j).$$

The complete definition of the Gromov convergence is quite sophisticated, and we omit it. An illustrating example will be given below.
Using the compactness theorem, Gromov established the following important result.

Proposition 2.1 (Persistence Principle)**.** Consider a "generic" family $g_s(z, w),\ s \in [0, 1]$ with $g_0 = 0$. Then either $P(0, g_s)$ has a solution for all s or bubbling off occurs at some $s_\infty \in [0, 1]$, i.e. there exists a subsequence $s_j \to s_\infty$ such that the sequence of solutions of $P(0, g_{s_j})$ converges to a cusp solution of $P(0, g_{s_\infty})$.

The word "generic" should be interpreted as follows. One can endow the space of all families g_s with an appropriate Banach manifold structure. Generic families form a residual subset (that is a countable intersection of open and dense subsets) in this space. In particular, every family g_s becomes generic after an arbitrarily small perturbation.

3 An application to the Liouville class

Definition 3.1. Let $L \subset (\mathbb{R}^{2n}, \mathrm{d}p \wedge \mathrm{d}q)$ be a Lagrangian submanifold. Consider the restriction $\lambda|_{TL}$ of the Liouville form

$$\lambda = p_1 \mathrm{d}q_1 + \cdots + p_n \mathrm{d}q_n$$

to L. The cohomology class $\lambda_L \in H^1(L, \mathbb{R})$ of this closed 1-form is called the **Liouville class** of the Lagrangian submanifold L. A closed Lagrangian submanifold $L \subset (\mathbb{R}^{2n}, \omega)$ is called **rational** if $\lambda_L(H_1(L; \mathbb{Z})) \subset \mathbb{R}$ is a discrete subgroup. We will denote its positive generator by $\gamma(L)$.

Theorem 3.1. Let $L \subset B^2(r) \times \mathbb{C}^{n-1}$ be a closed rational Lagrangian submanifold. Then $\gamma(L) \leq \pi r^2$.

Proof. Suppose that $L \subset B^2(r) \times \mathbb{C}^{n-1}$ is a closed Lagrangian submanifold. Take

$$g(z, w) = (\sigma, 0, \ldots, 0) \in \mathbb{C}^n$$

for some $\sigma \in \mathbb{C}$.

Lemma 3.1. If $|\sigma| > r$ then $P(0, g)$ has no solutions.

Proof of lemma. Suppose that f is a solution of $P(0, g)$ and denote by ϕ its first (complex) coordinate. Thus

$$\frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} = 2\sigma.$$

Since $L \subset B^2(r) \times \mathbb{C}^{n-1}$, we have that $|\phi|_{\partial D}| \leq r$. By Green's theorem

$$2\pi\sigma = \int_D \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \, \mathrm{d}x\mathrm{d}y = \int_{S^1} \phi \, \mathrm{d}y - i\phi \, \mathrm{d}x.$$

Now we can write $x + iy = e^{2\pi i t}$ and $\mathrm{d}x + i\mathrm{d}y = 2\pi i e^{2\pi i t}\mathrm{d}t$, so $\mathrm{d}y - i\mathrm{d}x = 2\pi e^{2\pi i t}\mathrm{d}t$. Therefore

$$2\pi|\sigma| = 2\pi \left| \int_0^1 e^{2\pi i t} \phi(e^{2\pi i t}) \, \mathrm{d}t \right| \leq 2\pi r$$

and hence $|\sigma| \leq r$ and the theorem is true by contrapositive. $\square$

Take now any σ with $|\sigma| > r$ and apply the persistence principle to the family

$$g_s = (s\sigma, 0, \ldots, 0), \ s \in [0, 1].$$

The previous lemma tells us that there is no solution for $s = 1$, so we have that for a small perturbation of g_s bubbling off takes place. For the sake of simplicity we assume that it happens in g_s itself. The general argument goes through without changes (make estimates up to ε) and is left to the reader.

We have a sequence $s_n \to s_\infty \leq 1$ and a decomposition $0 = \alpha + \beta_1 + \cdots + \beta_k,\ \beta_j \neq 0$. Let f_n be the solutions of $P(0, g_{s_n}),\ f_\infty$ a solution of $P(\alpha, g_{s_\infty})$ and $h_1, \ldots, h_k$ holomorphic discs with $[h_j] = \beta_j$ satisfying

$$\mathrm{Area}(f_n) \longrightarrow \mathrm{Area}(f_\infty) + \sum_{j=1}^{k} \mathrm{Area}(h_j).$$

Applying both parts of proposition (1.1) and using the fact that the discs h_j are holomorphic ($\bar{\partial} h_j = 0$) we get that $\mathrm{Area}(h_j) = \omega(h_j) \geq \gamma(L)$. This inequality follows from the fact that $[h_j] = \beta_j \neq 0$. From proposition (1.1) (ii) we deduce that

$$\mathrm{Area}(f_\infty) \geq |\omega(f_\infty)| = \left| \sum_{j=1}^{k} \omega(h_j) \right| \geq \gamma(L).$$

Thus $\mathrm{Area}(f_\infty) + \sum_{j=1}^{k} \mathrm{Area}(h_j) \geq 2\gamma(L)$. On the other hand proposition (1.1) (i) implies

$$\mathrm{Area}(f_n) \leq 2\pi s_n^2 |\sigma|^2 \leq 2\pi |\sigma|^2.$$

We use here that $\omega(f_n) = 0$ (since $[f_n] = 0$) and $\bar{\partial} f_n = g_{s_n}$. Putting these two inequalities together we get $2\pi|\sigma|^2 \geq 2\gamma(L)$. This is true for all σ with $|\sigma| > r$ so we have $\pi r^2 \geq \gamma(L)$, which proves the theorem. $\square$

Theorem 3.2. Assume that $L \subset \mathbb{R}^{2n}$ is a closed Lagrangian submanifold. Then the cohomology class $\lambda_L \neq 0$ i.e. L is *not exact.*

Proof. Consider a closed Lagrangian submanifold $L \subset B^2(r) \times \mathbb{C}^{n-1}$. Then the lemma above implies that the problem

$$\begin{cases} \bar{\partial} f(z) = (s\sigma, 0, \ldots, 0), & |\sigma| > r \\ [f] = 0. \end{cases}$$

has no solution for $s = 1$. The persistence principle implies that bubbling off must take place. This means that there exists a non-zero class β_1 which is represented by a holomorphic disc h_1. Since $h_1 \neq$ constant, we get $\omega(h_1) > 0$. Since we found a disc in $\mathbb{C}^n$ spanned by $h_1(\partial D)$ which has non-zero symplectic area, we conclude that $\lambda_L \neq 0$. $\square$

4 An example

Let $L = \partial D \subset \mathbb{C}$ and let $\sigma = 1$. We wish to find all maps $f : D \to \mathbb{C}$ such that $f(\partial D) \subset \partial D$ and

$$\begin{cases} \bar{\partial} f(z, \bar{z}) = s \\ [f|_{\partial D}] = 0. \end{cases} \tag{1}$$

Since $\frac{\partial f}{\partial \bar z} = s$ then $f(z,\bar z) = s\bar z + u(z)$ where u is a holomorphic function on D. We claim that the function $\varphi := s + zu(z)$ is holomorphic, takes ∂D to ∂D and that $\varphi|_{\partial D}$ has degree 1. Indeed,

$$zf(z,\bar z) = s|z|^2 + zu(z)$$

so

$$|z||f(z,\bar z)| = \left|s|z|^2 + zu(z)\right|.$$

If $|z| = 1$ this reduces to $|f(z,\bar z)| = |\varphi|$ but $|f(z,\bar z)| = 1$. So φ is a holomorphic function taking ∂D to ∂D. Observe that $\deg f = 0$ and $\deg z = 1$ so the total degree, $\deg zf = 1$ and hence $\deg \varphi = 1$. All such holomorphic functions are known as isometries of the hyperbolic metric in the unit disc. They have the form

$$e^{i\theta}\frac{1-\bar\alpha z}{z-\alpha}$$

for $\theta \in \mathbb{R},\ |\alpha| > 1$. Thus $\varphi(z) = e^{i\theta}\frac{1-\bar\alpha z}{z-\alpha}$ so

$$zu(z) = \frac{e^{i\theta} + \alpha s - z(s + e^{i\theta}\bar\alpha)}{z-\alpha}.$$

Since u is holomorphic, it cannot have any poles, so $e^{i\theta} + \alpha s = 0$ and hence $\alpha = -\frac{e^{i\theta}}{s}$. Now $1 < |\alpha| = \left|\frac{1}{s}\right|$ implies that $s < 1$ and that there are no solutions of (1) for $s \geq 1$. So bubbling off must occur at $s = 1$. For the sake of simplicity we put $\theta = 0$. Then

$$u(z) = -\frac{s - \frac{1}{s}}{z + \frac{1}{s}} = \frac{1-s^2}{sz+1}$$

so

$$f_s(z,\bar z) = s\bar z + \frac{1-s^2}{sz+1}.$$

When $s \to 1$, $f_s(z,\bar z) \to \bar z$ for all $z \neq -1$ and this convergence is uniform outside every neighbourhood of -1. Consider the graphs of f_s in $D \times D \subset \mathbb{C}\times\mathbb{C}$. Set $w = f_s(z,\bar z)$ so

$$(w - s\bar z)(sz+1) = 1 - s^2.$$

When $s \to 1$ this equation goes to $(w - \bar z)(z+1) = 0$ and the graph becomes the union of two curves

$$w = \bar z,$$

$$z = -1.$$

Here $w = \bar z$ is the graph of f_∞ and $\{z = -1\}$ corresponds to the holomorphic disc with boundary on $\{-1\} \times L$. Projecting onto the w-coordinate we get bubbling off. Indeed, $f_\infty(z) = \bar z$ is a solution of $P(-a,1)$ where $a = \left[S^1\right]$ and the holomorphic disc $h(z) = z$ is a solution of $P(a,0)$.

In order to visualize the bubbling off phenomenon we restrict to the real axes. Consider the graphs of the corresponding functions $f_s(x) = sx + \frac{1-s^2}{sx+1}$ for $x \in [-1,1]$. We get the following picture. The graphs of f_s converge to the union of two curves, the graph of the real part of f_∞ and the segment $I = [-1, 1]$ which is the real part of the holomorphic disc $\{-1\} \times D$.