In this lecture we cover chapters 5 & 6 of the book "The geometry of the group of symplectic diffeomorphisms" by L. Polterovich.

Chapter 5: Linearisation of Hofer's Geometry

GOAL: Interpret Hofer's metric as distance between a point and a linear subspace.

5.1. Space of periodic Hamiltonians

$\mathcal{F} := \{ F: M \times \mathbb{R} \to \mathbb{R}$ smooth normalised Hamiltonians with $F(x,t) = F(x,t+1)\}$

Lemma 1: A flow $\{f_t\}_{t\in\mathbb{R}}$ generated by some $F \in \mathcal{F} \iff f_{t+1} = f_t f_1 \quad \forall t \in \mathbb{R}$

Proof of Lemma 1: Let $\frac{d}{dt} f_t = X_t \, f_t$

"$\Rightarrow$": If $F_t = F_{t+1}$, consider the initial value problem $\begin{cases} \frac{d}{dt} \psi_t = X_t \psi_t \\ \psi_0 = f_1 \end{cases}$

Then $\frac{d}{dt}(f_{t+1}) = X_{t+1}(f_{t+1}) = X_t(f_{t+1})$, $f_{0+1} = f_1$

$\frac{d}{dt}(f_t f_1) = X_t(f_t) f_1 = X_t(f_t f_1)$, $f_0 f_1 = \mathbb{1} f_1 = f_1$

But since the solution to the ODE is unique we get $f_{t+1} = f_t f_1$

"$\Leftarrow$": $\frac{d}{dt}(f_{t+1}) = X_{t+1}(f_{t+1})$

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$\frac{d}{dt}(f_t f_1) = X_t(f_t) f_1 = X_t(f_t f_1) \Rightarrow X_{t+1} = X_t$, hence $F_t = F_{t+1}$

Remark: Take any flow $\{g_t\}_{t\in\mathbb{R}}$ with $g_1 = \phi$
Let $\alpha: [0,1] \to [0,1]$ with $\alpha \equiv 0$ near 0, $\alpha \equiv 1$ near 1
Consider new flow $f_t = g_{\alpha(t)}$ $t \in [0,1]$, extend by $f_{t+1} = f_t f_1$
By Lemma 1, $\{f_t\}_t$ is generated by some function in $\mathcal{F}$

We define a norm on $\mathcal{F}$ by $\quad |||F||| = \max_t \|F_t\| = \max_t (\max_x F(x,t) - \min F(x,t))$

Notation: $F \in \mathcal{F} \rightsquigarrow$ Hamiltonian flow $\{f_t\}_{t\in\mathbb{R}} \rightsquigarrow \phi_F := f_1$

Let $\mathcal{H} = \{F \in \mathcal{F} : \phi_F = \mathbb{1}\} \subset \mathcal{F}$. $\mathcal{H}$ generates Hamiltonian loops.

Theorem 1: $\forall F \in \mathcal{F}$: $\rho(\mathbb{1}, \phi_F) = \inf_{H\in\mathcal{H}} |||F - H||| = dist(F, \mathcal{H})$

For the proof of this theorem we need the following lemma:

Lemma 2: $\forall \phi \in Ham(M)$ we have $\rho(\mathbb{1}, \phi) = \inf\{ |||F||| : F \in \mathcal{F}$ s.t. $\phi = \phi_F\} =: r(\mathbb{1}, \phi)$

Proof of Thm 1 using Lemma 2:

For $F \in \mathcal{F}$ let $\{f_t\}_t$ its Hamiltonian flow
Let $\{g_t\}_t$ be another flow generated by $G \in \mathcal{F}$ with $g_1 = \phi_F$

(1) Decompose $g_t = h_t \circ f_t$ and H the normalised Hamiltonian of $\{h_t\}_{t\in\mathbb{R}}$

By Lemma 1: $H \in \mathcal{F}$ and $\{h_t\}_{t\in\mathbb{R}}$ is a loop

$\ulcorner$ $g_0 = f_0 = 1\!\!1,\ g_1 = f_1 = \phi_F \Rightarrow h_0 = h_1 = 1\!\!1$ $\lrcorner$

(2) Also $\forall$ loop $\{h_t\}_t$, then $\{h_t f_t\}_{t\in\mathbb{R}}$ is generated by some function in $\mathcal{F}$ using Lemma 1.

$\ulcorner$ $g_t := h_t f_t$, then $g_{t+1} = h_{t+1} f_{t+1} = h_t \underbrace{h_1}_{=1\!\!1} f_t f_1 = (h_t f_t)(h_1 f_1) = g_t g_1$ $\lrcorner$

Now in any case: $G(x,t) = H(x,t) + F(h_t^{-1}x, t)$

Then $H'(x,t) = -H(h_t x, t)$ generates the loop $\{h_t^{-1}\}_{t\in\mathbb{R}}$, hence $H' \in \mathcal{F}$

Therefore $|||G||| = |||F - H'|||$.

By (1), (2) every G corresponds to a unique H' and vice versa, hence we conclude by Lemma 2.

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Proof of Lemma 2:

Clearly $r(1\!\!1, \phi) \geq \rho(1\!\!1, \phi)$.

For $\varepsilon > 0$, let $\{f_t\}_{t\in[0,1]}$ Hamiltonian flow with $f_0 = 1\!\!1, f_1 = \phi$ & $\int_0^1 \|F_t\|\, dt \leq \rho(1\!\!1,\phi) + \varepsilon$

W.l.o.g let $F \in \mathcal{F}$ (by the first remark above) and $(*)$ $\|F_t\| > 0$ $\forall t \in [0,1]$

Let $\mathcal{C} = \{C^1\text{-smooth, orientation preserving diffeos of } S^1 = \mathbb{R}/\mathbb{Z} \text{ fixing } 0\}$

For $\alpha \in \mathcal{C}$, the path $\{f_{\alpha(t)}\}_{t\in\mathbb{R}}$ generated by $F^\alpha(x,t) = \alpha' F(x, \alpha(t))$

Consider $b(t) = \dfrac{\int_0^t \|F_s\|\, ds}{\int_0^1 \|F_s\|\, ds}$, let $\alpha := b^{-1}$.

$$|||F^\alpha||| = \max_t \alpha'(t) \|F_{\alpha(t)}\| = \max_t \left(\frac{\|F_t\|}{b'(t)}\right) = \int_0^1 \|F_s\|\, ds \leq \rho(1\!\!1, \phi) + \varepsilon$$

Approximating α by a smooth diffeo from $\mathcal{C}$, we get $\tilde{F} \in \mathcal{F}$ generating ϕ s.t.

$$|||\tilde{F}||| \leq \rho(1\!\!1, \phi) + 2\varepsilon$$

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Justification for (*):

Prop 1: $\{f_t\}_{t\in\mathbb{R}}$ flow generated by F, then $\exists$ arbitrary small (in C^∞-sense) loop $\{h_t\}_{t\in\mathbb{R}}$ s.t. $\{h_t^{-1}f_t\}_{t\in\mathbb{R}}$ is regular, i.e. $\exists x\in M$ s.t. $-H(h_t x, t)+F(h_t x, t)\neq 0$

Proof: Let $y\in M$, $E:=T_y^*M$. Take $2n$- smooth closed curves $\alpha_1,\dots,\alpha_{2n}$ s.t.
(i) $\{\alpha_1(t),\dots,\alpha_{2n}(t)\}$ is a basis $\forall t$ and (ii) $\int_0^1 \alpha_j(t)\,dt=0$ $\forall j$

Let $u_1, v_1,\dots,u_n,v_n$ be a basis of E take $\alpha_{2j+1}(t)=u_j\cos(2\pi t)+v_j\sin(2\pi t)$
$\alpha_{2j+1}(t)=-u_j\sin(2\pi t)+v_j\cos(2\pi t)$

Claim: For functions $G^{(1)},\dots,G^{(2n)}\in\mathcal{F}$ with $\int^1 G^{(j)}(x,t)\,dt=0$ & $d_yG_t^{(j)}=\alpha_j(t)$ there is a corresponding $2n$-dim variation $\{h_t(\varepsilon)\}$ for ε in a small neighborhood of 0 in $\mathbb{R}^n$ of the constant loop $h_t(0)=\mathbb{1}$ s.t. $\frac{\partial}{\partial\varepsilon_j}\big|_{\varepsilon=0}H(x,t,\varepsilon)=G^{(j)}(x,t)$ for $H(x,t,\varepsilon)$ the Hamiltonian of the variation.

Consider $\Phi: S^1\times\mathbb{R}^{2n}\longrightarrow E$, $(t,\varepsilon)\longmapsto d_y(F_t-H_t(\varepsilon))$

Since $\frac{\partial}{\partial\varepsilon_j}\big|_{\varepsilon=0}\Phi(t,\varepsilon)=\alpha_j(t)$, Φ is a submersion in some neighborhood U of $\varepsilon=0$

Let $\Psi=\Phi|_{S^1\times U}$, Ψ is a submersion, $\Psi^{-1}(\{0\})$ is a 1-dim submanifold of $S^1\times U$, hence its projection to U is nowhere dense

$\Rightarrow\exists$ arbitrary small $\varepsilon\in\mathbb{R}^{2n}$ s.t. $d_y(F_t-H_t(\varepsilon))\neq 0$ which proves the Proposition ▨

Construction in the claim:

First take the 1-dimensional case

Take $G\in\mathcal{F}$ s.t. $\int_0^1 G(x,t)\,dt=0$ $\forall x\in M$

Then let $h_t(\varepsilon)\in\mathrm{Ham}(M,\Omega)$ be the time ε map of the Hamiltonian flow generated by the time-independent Hamiltonian $\int_0^t G(x,s)\,ds$

Then for the corresponding Hamiltonian $H(x,t,\varepsilon)$ we have $\frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}H(x,t,\varepsilon)=G(x,t)$

Then for function $G^{(1)},\dots,G^{(2n)}\in\mathcal{F}$ with $\int_0^1 G^{(j)}(x,t)\,dt=0$, one can construct for every j a 1-dim variation $h_t^{(j)}(\varepsilon)$ as above.

Then $h_t(\varepsilon_1,\dots,\varepsilon_{2n})=h_t^{(1)}(\varepsilon_1)\circ h_t^{(2)}(\varepsilon_2)\circ\dots\circ h_t^{(2n)}(\varepsilon_{2n})$ is a $2n$-dim variation of the constant loop with
$$\frac{\partial}{\partial\varepsilon_j}\Big|_{\varepsilon=0}H(x,t,\varepsilon)=G^{(j)}(x,t)$$

Paths in Homotopy class

For $F \in \mathcal{F}$, $\{f_t\}_{t\in\mathbb{R}}$ the corresponding Hamiltonian. Consider

$$\ell(F) = \inf\{\, \text{length}(\{g_t\}_t) : \{g_t\}_{t\in[0,1]} \text{ Ham. path with } g_0 = \mathbb{1}, g_1 = \phi_F, \{g_t\}_t \simeq \{f_t\}\}$$

where $\text{length}(\{g_t\}_t) = \int_0^1 \|G_s\| ds$

<u>Remark:</u> $\ell(F)$ can be identified with $\tilde{\rho}(\tilde{\mathbb{1}}, \tilde{\phi}_F)$ the distance on the universal cover of $Ham(M,\Omega)$.

Let $\mathcal{H}_c \subset \mathcal{H}$ be the set of Hamiltonians that generate contractible loops or $\mathcal{H}_c$ is the path connected component of $\mathcal{H}$ containing 0.

<u>Theorem 2:</u> $\forall F \in \mathcal{F}$: $\ell(F) = \inf\limits_{H\in\mathcal{H}_c} |||F - H|||$

The proof of this theorem proceeds in the same way, just note that:

- Time reparametrizations & Regularisation doesn't change the homotopy class of a path with fixed end points. Hence it suffices to prove that
$$\ell(F) = \inf\{\, |||G||| : G \in \mathcal{F} \text{ with } \phi_G = \phi_F \text{ with Ham. flow of } G \text{ homotopic to } \{f_t\}_t\}$$
- $g_t = h_t \circ f_t$ & g_t is homotopic to f_t $\Rightarrow$ h_t is contractible

Chapter 6: Lagrangian intersections

6.1 Exact Lagrangian isotopies

(V^{2n}, ω) symplectic manifold, N^n closed manifold.

Let $\Phi : N \times [0,1] \longrightarrow V$ be a Lagrangian isotopy, i.e. $\{\Phi_t = \Phi(\cdot, t)\}_{t\in[0,1]}$ is a smooth family of Lagrangian embeddings.

$\Phi^*\omega = \alpha_s \wedge ds$ for $\{\alpha_s\}_s$ a family of 1-forms on N since $\Phi^*\omega|_{N\times\{t_0\}} \equiv 0$

Also $d\Phi^*\omega = d\alpha_s \wedge ds \equiv 0 \Rightarrow \alpha_s$ is closed $\forall s$

Def: Φ is exact if α_s is exact $\forall s$

Let $\{h_{t,s}\}_{t\in S^1, s\in[0,1]}$ be a smooth family of Hamiltonian loops of (M,Ω) generated by $H(x,t,s)$
$L \subset M$ a closed Lagrangian submanifold. Then

$\Phi : L \times S^1 \times [0,1] \longrightarrow M \times T^*S^1$ is the corresponding family of Lagrangian suspensions
$(x,t,s) \longmapsto (h_{t,s}x, -H(h_{t,s}x, t, s))$

Theorem 3: Φ is exact.

Proof: Write $\Phi^*(\Omega + dr \wedge dt) = \alpha_s \wedge ds$
Hence we need to check that α_s is exact

As in Chapter 3.1.E one can find that for $x \in L$, $\xi \in T_xL$:

$$\alpha_s(\xi) = \Omega\left((h_{t,s})_*\xi, \frac{\partial h_{t,s}}{\partial s}x\right) \quad ; \quad \alpha_s\left(\frac{\partial}{\partial t}\right) = \frac{\partial H}{\partial s}(h_{t,s}x, t, s)$$

Elements in $H_1(L\times S^1, \mathbb{Z})$ are generated by elements of the form $C = \beta \times \{0\}$, $D = \{y\}\times S^1$ (Algebraic Künneth Formula) for β a cycle on L, $y \in L$

Hence we need to prove that for all 1-cycles their integral vanishes

For cycles like C above we have that since $h_{0,s} = \mathbb{1}$ that α_s vanishes on all tangent vectors to $L \times \{r = 0\}$

Also $\int_D \alpha_s = \int_0^1 \frac{\partial H}{\partial s}(h_{t,s}y, t, s)\, dt \overset{(*)}{=} 0$

For $(*)$ one proves that $\frac{\partial}{\partial s} H(h_{t,s}x, t, s)$ is the derivative of a periodic function w.r.t. t. ▨

see Prop 6.1C in the book for the proof of $(*)$

6.2 Lagrangian Intersections

Def: A Lagrangian submanifold $N \subset V$ has the Lagrangian intersection property (LIP) if
$N \cap \Phi(N) \neq \emptyset \quad \forall \Phi \in \mathrm{Ham}(M,\omega) \iff e(N) = +\infty$

Example: Gromov's Theorem
- If $\pi_2(V,N)=0$ & V has "nice" behaviour at infinity, then N has this property.
- If $\pi_2(V,N) \neq 0$, take e.g. $V = S^2$, N a small enough circle, then N can be displaced

Def: A closed Lagrangian submanifold L of (M,Ω) has the stable Lagrangian intersection property if $L \times \{r=0\}$ has the LIP in $(M \times T^*S^1, \Omega + dr \wedge dt)$

Examples:
- Lagrangian tori in the cotangent bundle $T^*\mathbb{T}^n$ with standard symmetric structure (by Gromov's theorem)
- equator of S^2

6.3 Application to Hamiltonian loops

(M,Ω), $L \subset M$ closed Lagrangian submanifold that has the LIP
$\{g_t\}_{t\in S^1}$ loop of Hamiltonian diffeos generated by $G \in \mathcal{H}$
s.t. $g_t(L) = L \ \forall t \in S^1$; $G(x,t)=0 \ \forall x \in L, t \in S^1$

Theorem 4: $\{h_t\}_{t\in S^1}$ loop of Ham. diffeomorphisms homotopic to $\{g_t\}_{t\in S^1}$ with Hamiltonian $H \in \mathcal{H}$.
Then $\exists (x,t) \in L \times S^1$ s.t. $H(x,t) = 0$

Proof: By applying twice the Lagrangian suspension construction, we obtain two Lagrangian submanifolds N_G & N_H of $M \times T^*S^1$

Then $N_G = L \times \{r=0\}$ and N_H is isotopic to N_G by Thm 3 & the isotopy is exact

The stable LIP yields $N_H \cap N_G \neq \emptyset$, let $(x,0,t) \in N_H \cap N_G$

Since it lies in N_H: $\quad x = h_t y$ & $0 = -H(h_t y, t)$ for some $y \in L$ $\boxtimes$

Cor: If $L \subset M$ has the stable LIP, then $\forall H \in \mathcal{H}_c$ there is a $x \in L, t \in S^1$ s.t. $H(x,t)=0$

Remarks:
- Statement in the Corollary is not true if the (stable) LIP is not assumed
- Theorem 4 allows us to give non-trivial lower bounds for Hofer's distance.