The diameter of Hofer's group
Seminar: Introduction to Hofer's Geometry

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Goal

There is an open conjecture that for any symplectic manifold, the group $\mathrm{Ham}(M, \Omega)$ of Hamiltonian diffeomorphisms has infinite diameter with respect to Hofer's metric

$$\rho(\mathbb{1}, \phi) = \inf \operatorname{length}\{f_t\}$$

where the infimum is taken over all Hamiltonian paths $\{f_t\}$ with $f_0 = \mathbb{1}$ and $\phi_F := f_1 = \phi$. Today, we will prove the conjecture for closed orientable surfaces.

Notation and Results

We start by recalling some of the notation and results from previous presentations. The theorem numbering refers to the numbering from [Pol01].
We denote by $\mathcal{F}$ the space of smooth normalized *periodic* Hamiltonian funcitons $F : M \times \mathbb{R} \to \mathbb{R}$. Every $\phi \in \mathrm{Ham}(M, \Omega)$ is the time-one flow diffeomorphism of a flow induced by a Hamiltonian $F \in \mathcal{F}$. $\mathcal{H} \subseteq F$ is the set of Hamiltonian that generate loops in $\mathrm{Ham}(M, \Omega)$, with $\mathcal{H}_c \subseteq \mathcal{H}$ consisting of those that generate contractible loops. Here, contractible means that there exists a homotopy $[0, 1] \times [0, 1] \to \mathrm{Ham}(M, \Omega)$ from $\{f_t\}$ to the constant loop at $\mathbb{1}$ with respect to the strong Whitney C^∞-topology. We denote by $\mathrm{Diff}_0(M)$ the path connected component of diffeomorphisms of the surface M, and by $\mathrm{Symp}_0(M, \Omega)$ the path connected componenent of symplectic diffeomorphisms of M.
For $F \in \mathcal{F}$, we define its norm and length by

$$|||F||| = \max_t \left(\max_x F(x, t) - \min_x F(x, t) \right) \quad \text{and} \quad l(F) = \inf \operatorname{length}\{g_t\}$$

where the infimum is taken over all Hamiltonian flows $\{g_t\}$ which are homotopic to $\{f_t\}$ rel endpoints.
Last week, we have seen a characterization of the norm and the length.

Theorem (5.1.B/5.3.A)**.** *For every $F \in \mathcal{F}$*

$$\rho(1, \phi_F) = \inf_{H \in \mathcal{H}} |||F - H||| \quad \textit{and} \quad l(F) = \inf_{H \in H_c} |||F - H|||$$

Note that if $\mathrm{Ham}(M, \Omega)$ is simply connected, then $\rho(\mathbb{1}, \phi_F) = l(F)$.

Starting Estimate

We begin by obtaining a bound for $l(F)$.

Proposition (7.1.A)**.** *Let $L \subseteq M$ be a closed Lagrangian submanifold which has the stable Lagrangian intersection property. Let $F \in \mathcal{F}, C > 0$ such that*

$$|F(x,t)| \geq C \quad \textit{for all} \quad x \in L, t \in S^1$$

then $l(F) \geq C$.

Proof. Let $H \in \mathcal{H}_c$. By Corollary 6.3.B, there exist $y \in L, \tau \in S^1$ such that $H(y, \tau) = 0$. Moreover, since $F - H$ is normalized for all t, let $z \in M$ be such that $F(z, \tau) - H(z, \tau) = 0$, then

$$\begin{aligned} |||F - H||| &= \max_t \|F_t - H_t\| \\ &\geq \max_{x_1} F(x_1, \tau) - H(x_1, \tau) - \min_{x_2} F(x_2, \tau) - H(x_2, \tau) \end{aligned}$$

If $F(x,t)$ is positive on $L \times S^1$, choose $(x_1, x_2) = (y, z)$. If $F(x,t)$ is negative, choose $(x_1, x_2) = (z, y)$. In either case we get $|||F - H||| \geq C$. Since this holds for all $H \in \mathcal{H}_c$, it follows

$$l(F) = \inf_{H \in \mathcal{H}_c} |||F - H||| \geq C$$

□

The fundamental group

We wish to compute the fundamental group $\pi_1(\mathrm{Ham}(M, \omega))$ in order to extend this estimate for $\rho(1, \phi_F)$. Unfortunately, not much is known about the group for general general symplectic manifolds. Even in the simple example of $\mathbb{R}^{2n}$, $\pi_1(\mathrm{Ham}(\mathbb{R}^{2n}))$ is not known for $n \geq 3$. For closed surfaces however, the fundamental group can be computed. Before we state the main theorem, we cover some preliminary results.
For a space X, denote by ι the inclusion

$$\iota : X \times \{0\} \to X \times [0, 1]$$

Definition. Let $p : E \to B$ be a continuous map. We say that p has the **Homotopy lifing property** with respect to a space X, if for any homotopy $h : X \times [0, 1] \to B$, such that $h_0 = h \circ \iota$ can be lifted to a map $\tilde{h}_0 : X \times \{0\} \to E$, there exists a lift $\tilde{h} : X \times [0, 1] \to E$ of h.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{h}_0} & E \\ \downarrow \iota & \overset{\tilde{h}}{\nearrow} & \downarrow p \\ X \times [0,1] & \xrightarrow{h} & B \end{array}$$

A **Fibration** is a map $p : E \to B$ satisfying the Homotopy Lifting property for all topological spaces X. A **Serre Fibration** is a map $p : E \to B$ satisfying the Homotopy Lifting property for all CW-complexes.

For example, the tangent bundle $TM \to M$ is a fibration. A covering is a fibration where the homotopy lift is unique.

Lemma. *Let B be the set of all area forms on M with total area 1. For $\omega \in X$ fixed, the map*

$$p : \mathrm{Diff}_0(M) \to B, \quad \phi \mapsto \phi^*\omega$$

is a Serre fibration.

Proof. Recall Moser's stability theorem. It states that for every family $\{\omega_t\}$ of symplectic forms ω_t with an exact derivative $\frac{d}{dt}\omega_t = d\sigma_t$, there exists a family of diffeomorphisms $\phi_t \in \mathrm{Diff}(M)$ such that

$$\phi_t^*\omega_t = \omega_0$$

In view of our definitions above, this is saying that the map p has the homotopy lifting property for one-point space $X = \{*\}$. By repeated application, one can show that the map has the homotopy lifting property for the spaces $[0, 1]^n$. Since CW-complexes can be built by gluing together spaces of the form $[0, 1]^n$, the result follows. □

Corollary (A)**.** *The map $\pi_1(\mathrm{Symp}_0(M, \Omega)) \to \pi_1(\mathrm{Diff}_0(M))$ induced by the inclusion is an isomorphism.*

Since the map $p : \mathrm{Diff}_0(M) \to B$ is a Serre fibration, there is a long exact sequence

$$\pi_{n+1}(B) \to \pi_n(F) \to \pi_1(\mathrm{Diff}_0(M)) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots \to \pi_0(\mathrm{Diff}_0(M))$$

where F is the fiber of the map p. But since that is just $\mathrm{Symp}_0(M)$ and because the base B is contractible, we obtain the desired isomorphism.

Lemma (B). *Let M be an oriented smooth closed surface.*

(a) If M is the sphere or projective plane, then $\mathrm{Diff}(M) = \mathrm{Diff}_0(M)$ has $\mathrm{SO}(3)$ as strong deformation retract.

(b) If M is the torus, then $\mathrm{Diff}_0(M)$ has $\mathbb{T}^2$ as strong deformation retract.

(c) If M has genus ≥ 2, then $\mathrm{Diff}_0(M)$ is contractible.

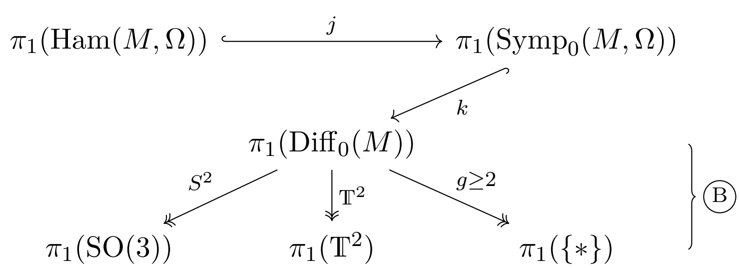
Here, the subset $\mathrm{SO}(3)$ acts by rotation of the sphere and $\mathbb{T}^2$ by translation on the torus. A proof of this is given in [EE43] (Corollary p.21).

Theorem (7.2.A/7.2.B). *Let (M, Ω) be a closed symplectic surface.*

(a) If $M = S^2$, the inclusion $\mathrm{SO}(3) \to \mathrm{Ham}(S^2)$ induces an isomorphism of fundamental groups.

(b) If M has genus $g \geq 1$, then $\pi_1(\mathrm{Ham}(M, \Omega)) = 0$.

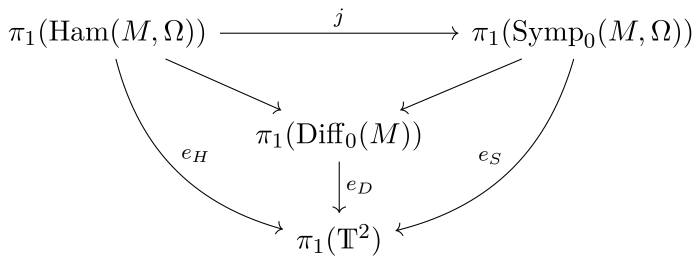
We give a sketch of the proof, which is structured as follows: Consider the diagram



where the maps j, k are induced by inclusion and the maps in (B) are chosen depending of the genus of the surface.
Corollary A shows that the map k is an isomorphism. Lemma B shows that the maps in (B) (depending on the genus) are isomorphism.
In [MS98] (see Proposition 10.18 (i)), it is shown that the map j is injective.
This proves part (a) of the theorem and (b) for the case $g \geq 2$.
It remains to handle the case of the torus. Fix a point $y \in \mathbb{T}^2$ and consider the evaluation map

$$e : \mathrm{Diff}_0(\mathbb{T}) \to \mathbb{T}^2, \quad f \mapsto f(y)$$

This induces a map $e_D : \pi_1(\mathrm{Diff}_0(\mathbb{T}^2)) \to \pi_1(\mathbb{T}^2)$. By Lemma B this is an isomorphism. Now consider the restrictions of e to $\mathrm{Ham}(\mathbb{T}^2)$ and $\mathrm{Symp}_0(\mathbb{T}^2)$ and write e_H, e_S for the respective induced group homomorphisms.



Since e_D, e_S are isos and j is injective, it follows from $e_H = j \circ e_S$ by a theorem from Floer that e_H vanishes (see [FP98]). But e_H is injective, which can only be zero if $\pi_1(\mathrm{Ham}(\mathbb{T}^2))$ is zero.

Theorem. *Let $F \in \mathcal{F}, C > 0$ be such that $|F(x,t)|$ for all $x \in L, t \in S^1$.*

(a) if M is a closed orientable surface of genus $g \geq 1$ and L is a non-contractible surface,

(b) If $M = S^2$ and $L \subseteq M$ is an equator, or

then $\rho(1, \phi_F) \geq C$.

Proof. (a) This follows directly from the previous theorem and Theorem 7.1.A.

(b) Since the fundamental group is generated by a 1-turn rotation, the Hamiltonian vanishes on L (see Example 6.3.C). By Corollary 6.3.A, every function from $\mathcal{H}$ vanishes at some point (x_0, t_0) of $L \times S^1$. Thus

$$\rho(1, \phi_F) = \inf_{H \in \mathcal{H}} |||F - H||| \geq C$$

□

Corollary (7.2.D)**.** *The group of Hamiltonian diffeomorphisms of a closed surface has inifinite diamter with respect to Hofer's metric.*

Proof. Let L be as in Theorem 7.2.C, and let $B \subset M$ be an open disc disjoint from L. Take a Hamiltonian $F \in \mathcal{F}$ which is identically to C outside B.
The theorem then implies $\rho(\mathbb{1}, \phi_F) \geq C$. By taking C arbitrarily large, we can put ϕ_F arbirarily far away from $\mathbb{1}$.

□

Note that we can shrink the ball B and increase C such that ϕ_F converges pointwise to $\mathbb{1}$, but diverges in Hofer's metric.

The length spectrum

Instead of hoping that $\pi_1(\mathrm{Ham}(M,\Omega))$ is trivial, we wish to find another way of providing an estimate for $\rho(\mathbb{1}, \phi_F)$ that works for a larger class of manifolds.

Definition (7.3.A)**.** For $\gamma \in \pi_1(\mathrm{Ham}(M,\Omega))$, define its **norm** by

$$\nu(\gamma) = \inf \mathrm{length}\{h_t\}$$

where the infimum goes over all Hamiltonian loops which represent γ. We define the **length spectrum** of $\mathrm{Ham}(M,\Omega)$ to be the set

$$\{\nu(\gamma) | \gamma \in \pi_1(\mathrm{Ham}(M,\Omega))\}$$

Remark. $\pi_1(\mathrm{Ham}(M,\Omega))$ is commutative. Indeed, if we write $\circ$ for the composition of paths $\circ$ modulo homotopy and $*$ for the point-wise multiplication in $\mathrm{Ham}(M,\Omega)$, then they satisfy

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

From this, it follows $\circ$ and $*$ coincide and are commutative. Thus write $+$ for addition in $\pi_1(\mathrm{Ham}(M,\Omega))$ and 0 for the neutral element.
It also holds

$$\nu(\gamma) = \nu(-\gamma) \quad \text{and} \quad \nu(\gamma + \gamma') \leq \nu(\gamma) + \nu(\gamma')$$

The first equation follows from the fact that the inverse loop is generated by the reverse path. The second equation holds because the path generated by point-wise multiplication of representing loops of γ, γ' in $\mathrm{Ham}(M,\omega)$ is generated by the sum of their Hamiltonians.

It is not known whether ν is non-degenerate or not, so ν is a priori really a pseudo-norm. The next definition lets us find another nice class of manifolds for which we can get a bound for the metric.

Definition. We say that an open symplectic manifold (M,Ω) has the **Liouville property**, if there exists a smooth family of diffeomorphisms

$$D_c : M \to M, \quad c \in (0,\infty)$$

such that $D_1 = \mathbb{1}$ and $D_c^*\Omega = c\Omega$.

For example, the cotangent bundle $\pi : T^*N \to N$ with diffeomorphisms D_c given fiberwise by $(p,q) \mapsto (cp,q)$ has the Liouville property.

Lemma. *If (M,Ω) has the Liouville property, then its length spectrum is $\{0\}$.*

Proof. Let $\{h_t\}$ be a loop of Hamiltonian diffeomorphisms and $\{D_c\}$ as above. Then for every $c > 0$, the flow $\{D_c h_t D_c^{-1}\}$ is generated by the Hamiltonian $cF(D_c^{-1}x,t)$. Thus the length goes to zero as $c \to 0$, meaning that $\{h_t\}$ is homotopic rel endpoints to a loop of arbitrarily small length. □

Refining the estimate

Theorem (7.4.A)**.** *Let (M, Ω) be a symplectic manifold and let $L \subseteq M$ be a closed Lagrangian submanifold with the stable Lagrangian intersection property.*
Assume that the length spectrum of $\mathrm{Ham}(M, \Omega)$ *is bounded from above by some $K \geq 0$. Let $F \in \mathcal{F}$ be such that $|F(x,t)| \geq C$ for all $x \in L$ and $t \in S^1$.*
Then

$$\rho(\mathbb{1}, \phi_F) \geq C - K$$

Proof. Let $\epsilon > 0$ and let $\{g_t\}$ be a path in $\mathrm{Ham}(M, \Omega)$ from $\mathbb{1}$ to ϕ_F. Consider the loop $\{f_t g_t^{-1}\}$ and let γ be its homotopy class. By definition of ν, there is a loop $\{h_t\}$ which is homotopic to $\{f_t g_t^{-1}\}$ which by assumption has length

$$\mathrm{length}\{h_t\} \leq \nu(\gamma) + \epsilon \leq K + \epsilon$$

By Proposition 7.1.A, we have $l(F) \geq C$ and since $\{h_t g_t\}$ is homotopic to $\{f_t\}$, we have

$$\begin{aligned} C &\leq l(F) \\ &= \inf_{\text{paths } \{s_t\} \text{ homotopic to } \{f_t\}} \mathrm{length}\{s_t\} \\ &\leq \mathrm{length}\{h_t g_t\} \\ &\leq \mathrm{length}\{h_t\} + \mathrm{length}\{g_t\} \\ &\leq K - \epsilon + \mathrm{length}\{g_t\} \end{aligned}$$

letting $\epsilon \to 0$, we get $\mathrm{length}\{g_t\} \geq C - K$. As $\{g_t\}$ was an arbitrary path joining $\mathbb{1}$ to ϕ_F we obtain $\rho(\mathbb{1}, \phi_F) \geq C - K$. □

References

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