GROWTH & ONE-PARAMETER SUBGPS OF $HAM(M,\Omega)$.

In the following, we will study the growth of one-parameter subgroups of Hamiltonian diffeomorphisms on some symplectic manifold (M,Ω). To that aim always consider time-independent Hamiltonian functions $F: M \to \mathbb{R}$ (unless otherwise stated). Note that this justifies our discussion around "one parameter subgroups" as in this context, the flow $t \mapsto f_t$ of the Hamiltonian F is simply the flow of the vector field sgrad F, whence $f_{t+s} = f_t \circ f_s$ for all $t,s \in \mathbb{R}$.

Our goal will be to understand the interplay between the dynamics of the flow $\{f_t\}$ and the growth of $\rho(\mathbb{1}, f_t)$. We will mainly focus on the bounds on the growth obtained from the dynamics. Our central result is theorem 1.3. To conclude, we shall adress some generalizations of our problematic and some open questions.

§1 Invariant tori of classical mechanics.

We start by studying the example of invariant tori in classical mechanics. These invariant tori of Hamiltonian dynamical systems play an important role in physics.

Consider the n-dimensional torus $\mathbb{T}^n$ endowed with the Euclidean metric $ds^2 = \sum_{j=1}^{n} dq_j^2$. By the Weinstein neighbourhood theorem, the Euclidean geodesic flow can be described by a Hamiltonian system on the cotangent bundle $T^*\mathbb{T}^n$ endowed with the standard symplectic form $\Omega = dp \wedge dq$. The Hamiltonian function is given by $F(p,q) = \frac{1}{2}|p|^2$ (which one can look at as kinetic energy). Solving the Hamiltonian system

$$\begin{cases} \dot{p} = 0 \\ \dot{q} = p \end{cases}$$

yields the Hamiltonian flow $f_t(p,q) = (p, q+pt)$.

What can we say about the dynamics of this flow? Every torus $\{p=a\}$ is invariant under $\{f_t\}$. Moreover, the restriction of f_t to such torus is a (quasi-) periodic motion $q \to q+at$. Further, the invariant tori are all homologous to the zero section of the cotangent bundle.

This example is an instance of a central class of Hamiltonian (time-independent) systems called integrable systems. These are characterized by the fact that their energy levels are foliated (up to measure zero) by invariant middle-dimensional tori, which carry (quasi-) periodic motion. The behaviour of such systems under perturbation is studied in the Kolmogorov-Arnold-Moser (KAM for short) theory. In particular, the theory has shown that in the case of small perturbations, if one assumes a to be "irrational" (in the sense that its components $\{a_i\}_{i=1}^n \subset \mathbb{R}$ are $\mathbb{Q}$-linearly independent) then most of the invariant tori persist.

This example is particularly interesting as we shall see (cf theorem 1.3) that invariant tori homologous to the zero section contribute to the linear growth of $\rho(\Phi, f_t)$. The assumption of irrationality of the "pent" a will also make its appearance in the discussion below.

Note that the invariant tori above are Lagrangian. Actually, this is a more general phenomenon, as we shall now see:

Prop^{n} 1.1 Let $F: M \to \mathbb{R}$ Hamiltonian on a symplectic mfd M. If a closed submfd $L \subset \{F = c\}$ is Lagrangian, then L is invariant under the flow of F.

Proof: We know that sgrad F is tangent to the level sets of F thus tangent to L. The result then follows immediately as the flow of F is nothing but the flow of the vector field sgrad F ∎

In the case of tori as in the discussion above, the converse also holds:

Prop^{n} 1.2 Consider a Hamiltonian function $F: T^*T^n \to \mathbb{R}$ and $L \subset \{F = c\}$ an invariant torus with quasi-periodic motion as earlier, with $\dot{\theta} = a$ irrational. Then L is Lagrangian.

Proof: We need to show $\Omega|_{TL} = 0$.

Pick $x \in L$ and write $\Omega|_{T_xL} = \sum b_{ij}\, d\theta_i \wedge d\theta_j$ for some b_{ij}. As $\theta: t \mapsto \theta(0) + at$ is just a shift, this expression is valid at every point on the trajectory of x. Since every trajectory is dense in the torus so $\Omega = \sum b_{ij}\, d\theta_i \wedge d\theta_j$ everywhere on L. Now $\Omega|_{TL}$ is an exact 2-form. Hence, we must have $b_{ij} = 0$ for all i,j and $\Omega|_{TL} = 0$. ∎

Before digging deeper into this and define asymptotic growth, let us define a quantity that can be thought of as lower bound for the growth of flows of norm-one Hamiltonian functions.

Def^{n} For F a compactly supported Hamiltonian on (T^*T^n, Ω) define $\underline{E(F)} := \sup\{E : \{F = E\}$ contains a Lagrangian torus homologous to the zero section$\}$

Then one can show:

Theorem 1.3: $\rho(\mathbb{1}, f_t) \geq t \cdot E(F) \quad \forall t \in \mathbb{R}$ ✪

For $\{f_t\}$ the flow of F.

Proof: WLOG $t=1$ by time reparametrization.

Let L be a Lagrangian torus homologus to the zero section contained in some $\{F=E\}$ be arbitrary. Then we have shown that L has the stable Lagrangian intersection property. Also, since $T^*\mathbb{T}^n$ is Liouville, the length spectrum of $\mathrm{Ham}(T^*\mathbb{T}^n, \Omega)$ is $\{0\}$. Hence $\rho(\mathbb{1}, f_1) \geq E$. Conclude by taking the supremum over all E for which such L exists ∎

§2 Growth of one-parameter subgroups

Consider the following situation: For a symplectic manifold (M, Ω) and consider the flow $\{f_t\}$ of some normalized Hamiltonian function $F \in \mathcal{A}$. As mentionned earlier, we are interested in the central questions in Hofer's geometry, namely the relation between $t \mapsto \rho(\mathbb{1}, f_t)$ and the dynamics of the flow $\{f_t\}$. For example, we have seen in theorem 1.3 that invariant tori of an autonomous (= time independent) Hamiltonian flow homologous to the zero section contribute to the linear growth of $\rho(\mathbb{1}, f_t)$. Another reason for our interest is the link to the theory of geodesics in Hofer's metric, to which we now briefly turn our attention.

Def: A Hamiltonian path $\{f_t\}$ is a strictly minimal geodesic if each of its segments minimizes the length between its endpoints.

Conjecture: All one-parameter subgroups are locally strictly minimal geodesics (i.e all sufficiently short segments of one-parameter subgroups are strictly minimal geodesics).

Def: We define the asymptotic growth of $\{f_t\}$ generated by $\|F\|$ as

$$\mu(F) = \lim_{t \to \infty} \frac{\rho(\mathbb{1}, f_t)}{t \cdot \|F\|}$$

Rmk: Using that F is autonomous and ρ subadditive, one sees that $\mu(F) \in [0,1]$. In particular, it is well-defined.

Fact: If $\mu(F) < 1$, then $\{f_t\}$ is not a strictly minimal geodesic

Rmk: If $\|F\|=1$, then $E(F) \leq \mu(F)$ from thm 1.3.

The conjecture has been established for one-parameter subgroups of $\mathrm{Ham}(\mathbb{R}^{2n})$ by Hofer himself [Hof93]. Since then, it was extended to other symplectic manifolds including cotangent bundles, closed oriented surfaces and $\mathbb{C}P^2$ [LM95]. On the other hand, Sikorav [Sik90] discovered the striking fact that any such $\{f_t\} \subset \mathrm{Ham}(\mathbb{R}^{2n})$ remains at a bounded distance of identity, and thus cannot be globally strictly minimal.

Thm 2.1: Let $\{f_t\} \subset \mathrm{Ham}(\mathbb{R}^{2n})$ be a one-parameter subgroup generated by a Hamiltonian function F s.t. $\mathrm{supp}(F)$ is contained in a Euclidean ball of radius r. Then $\rho(\mathbb{1}, f_t) \leq 16\pi r^2$.

Proof: See [HZ94, p177]

Let us now return to the case of the torus. In [LM95], Lalonde and McDuff show that every $\{f_t\} \subset \mathrm{Ham}(T^*T^n)$ is locally strictly minimal. This implies that $\rho(\mathbb{1}, f_t) = t\|F\|$ for small t, so our estimate $(*)$ in theorem 1.3 is not sharp in that case. However, it is in the following case:

Thm 2.2: Let $F \geq 0$ be a compactly supported Hamiltonian on the cylinder T^*T^1 with $\|F\|=1$. Then $E(F) = \mu(F)$.

Rmk: The theorem then gives the sharpness as $\rho(\mathbb{1}, f_t) \overset{(*)}{=} t\cdot E(F) = t \lim_{s\to\infty} \frac{\rho(\mathbb{1}, f_s)}{s}$

Proof: We start by observing that $E(F) \leq \mu(F)$ by a previous remark. Remains to show $E(F) \geq \mu(F)$. Hence assume $E(F) < 1$.

The idea is to decompose $\{f_t\}$ into a product of commuting flows with simpler asymptotic behaviour. Pick $\varepsilon > 0$ and $u: [0,\infty) \to [0,\infty)$ smooth satisfying

$$u(s) = \begin{cases} s & \text{for } s \leq E(F)+\varepsilon \\ E(F)+2\varepsilon & \text{for } s \geq E(F)+3\varepsilon \end{cases} \quad \& \quad u(s) \leq s \text{ for all } s.$$

Consider the Hamiltonians $G = u \circ F$ and $H = G - F$, with respective flows $\{g_t\}$ and $\{h_t\}$. These flows commute and satisfy $f_t = g_t h_t$, whence $\rho(\mathbb{1}, f_t) \leq \rho(\mathbb{1}, g_t) + \rho(\mathbb{1}, h_t)$.

Note that $\|G\| \leq E(F) + 2\varepsilon$ and so $\rho(\mathbb{1}, g_t) \leq t(E(F) + 2\varepsilon)$

On the other hand, the support of H is contained in $D_\varepsilon = \{F \geq E(F) + \varepsilon\}$. For ε small enough, ∂D_ε consists of contractible closed curves by the definition of $E(F)$. Let $a > 0$ such that $\mathrm{supp}(F)$ is contained in an annulus $A = \{(p,q) \in T^*\mathbb{T}^1 : |q| \leq \frac{a}{2}\}$. Note that $\partial D_\varepsilon \subset A$. Hence D_ε is contained in some $D' \subset A$ which is a finite union of pw. disjoint closed discs of total area a. From the Dacorogna-Moser theorem [HZ94], since $T^*\mathbb{T}^1$ has infinite area, there exists a symplectic embedding $i : \mathbb{R}^2 \to T^*\mathbb{T}^1$ and a finite union of Euclidean discs $D'' \subset \mathbb{R}^2$ mapped diffeomorphically by i onto D'. Clearly, i induces a natural homomorphism $i_* : \mathrm{Ham}(\mathbb{R}^2) \to \mathrm{Ham}(T^*\mathbb{T}^1)$ which does not increase the Hofer metric. Then since $\{h_t\}$ is in the image of i_*, by theorem 2.1, $\rho(\mathbb{1}, h_t) \leq 16a$.

Combining everything, we get

$$\rho(\mathbb{1}, f_t) \leq t(E(F) + 2\varepsilon) + 16a \qquad (+)$$

for all $t > 0$, which implies $\mu(F) \leq E(F)$ by dividing by t, and letting $t \to +\infty$, $\varepsilon \to 0$. ∎

Rmk: • The same proof shows $\mu(F) = 0 \Rightarrow E(F) = 0 \overset{(+)}{\Rightarrow} \rho(\mathbb{1}, f_t)$ bounded.

• The same reasoning (& theorem) holds for any open surface of infinite area.

• The result can be extended to not-necessarily non-negative F.

However no generalization of thm 2.2 to higher dimensions is known. There is yet still a special case where (*) is sharp which we shall now adress.

Example:

Consider $F: T^*\mathbb{T}^1 \to \mathbb{R}$ a compactly supported Hamiltonian satisfying

(1) $F \geq 0$

(2) $\max F = 1$

(3) the maximum set $\Sigma = \{F = 1\}$ is a smooth section of the cotangent bundle.

What can one say about the geometry of the associated flow $\{f_t\}$? This turns out to drastically depend on the nature of Σ!

Case I: If Σ is Lagrangian, then $E(F) = 1$ by definition, whence $\{f_t\}$ is a locally strictly minimal geodesic, $E(F) = \nu(F)$.

Case II: If Σ is not Lagrangian, it is not known whether $\nu(F)$ equals $E(F)$. What is known, is that in this case, ⊛ gives a non trivial estimate in the sense that $\nu(F) < 1$. (*)

This follows from:

Thm 2.3 Let $F: M \to \mathbb{R}$ be a normalized Hamiltonian. Let Σ denote either its maximum or minimum set, and suppose there exists $\Phi \in Ham(M, \Omega)$ so that $\Phi(\Sigma) \cap \Sigma = \emptyset$. Then $\nu(F) < 1$. (In particular, the flow of F is not a strictly minimal geodesic).

Indeed, using Gromov's h-principles for partial differential equations we can show (see eg. [Pol 95]) the existence of a ϕ as in theorem 2.3 in the case where Σ is not Lagrangian.

§3 "curve shortenings - the proof of 2.3.

In this section, we shall prove theorem 2.3. We start by introducing

Def^n For $0 \neq F \in \mathcal{A}$, define $\delta(F) = \inf\left\{ \frac{\|F + F\circ\Phi\|}{2\|F\|} : \Phi \in Ham(M, \Omega)\right\}$

Thm 3.1: $\nu(F) \leq S(F)$.

Rmk: In the setting of theorem 2.3, one clearly has $S(F) < 1$, whence 2.3 will easily follow from theorem 3.1.

Proof: WLOG assume $\|F\| = 1$. Pick $\Phi \in Ham(M, \Omega)$, $T > 0$ arbitrary and write:

$$f_{2T} = (f_T \circ \Phi \circ f_T \circ \Phi^{-1})(\Phi \circ f_T^{-1} \circ \Phi^{-1} \circ f_T) =: A \cdot B.$$

As B is a commutator, we know $\rho(\mathbb{1}, B) \leq 2\rho(\mathbb{1}, \Phi)$.

Claim: $\rho(\mathbb{1}, A) \leq T \|F + F \circ \Phi^{-1}\|$.

Why? Let $g_t = f_t \circ \Phi \circ f_t \circ \Phi^{-1}$ for $t \in [0, T]$ be the path generating the diffeomorphism A. The corresponding Hamiltonian is $G(x,t) = F(x) + F(\Phi^{-1} f_t^{-1} x)$ whence:

$$\begin{aligned} \|G_t\| &= \|F + F \circ \Phi^{-1} \circ f_t^{-1}\| \\ &= \|F \circ f_t^{-1} + F \circ \Phi^{-1}\| \\ &= \|F + F \circ \Phi^{-1}\| \end{aligned}$$

which shows the claim.

Finally we obtain:

$$\frac{\rho(\mathbb{1}, f_{2T})}{2T} \leq \frac{\|F + F \circ \Phi^{-1}\|}{2} + \frac{\rho(\mathbb{1}, \Phi)}{T}$$

for all $\Phi \in Ham(M, \Omega)$. Letting T go to infinity, we deduce $\nu(F) \leq S(F)$ ∎

§4 General case

We have seen that on the cylinder (and any open surface of infinite area), one-parameter group whose asymptotic growth vanishes remains at bounded distance of identity. What can we say about other symplectic manifolds. The main raised question is

Q: Is there a symplectic manifold (M, Ω) with one-parameter subgroup $\{f_t\} \subset Ham(M, \Omega)$ such that $\rho(\mathbb{1}, f_t)$ has sublinear non-trivial growth at infinity (f. ex. $\sqrt{t}$)?

As it turns out, we don't know! This question remains an open problem, even for the two-torus $M = \mathbb{T}^2$. This can be justified by the fact that Hamiltonian functions $F \in \mathcal{A}(\mathbb{T}^2)$ inducing flows with zero asymptotic growth are "non-generic", in the following sense:

Thm 4.1: If 0 is a regular value of $F \in \mathcal{A}(\mathbb{T}^2)$, then $\mu(F) > 0$.

Proof: Consider $D = \{F = 0\}$. Since it is a finite number of pairwise disjoint embedded circles, there exists a non-contractible simple closed curve $L \subset \mathbb{T}^2$ satisfying $L \cap D = \emptyset$. Hence, there is $C > 0$ such that $|F(x)| > C$ $\forall x \in L$. Now $\pi_1(\mathrm{Ham}(\mathbb{T}^2)) = 0$ so by a previous result from this seminar, $\rho(\mathbb{1}, f_t) \geq Ct$. $\forall t$ ■

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