Symplectic fibrations

401-4530-23L Introduction to Hofer's Geometry

10.05.2023 FS23

Motivation

When looking at the diameter of Hofer's norm we defined the norm of $\gamma \in \pi_1(\mathrm{Ham}(M,\Omega))$ as

$$\nu(\gamma) = \inf_F \mathrm{length}\{f_t\} = \inf_F \int_0^1 \max_x F_t - \min_x F_t dt$$

where F_t is the corresponding Hamiltonian and the infimum goes over all normalized periodic Hamiltonian functions generating a loop representing γ. We then defined the length spectrum of $\mathrm{Ham}(M,\Omega)$ as

$$\{\nu(\gamma) \mid \gamma \in \pi_1(\mathrm{Ham}(M,\Omega))\}$$

We saw that $\nu(\gamma) = \nu(-\gamma)$ and $\nu(\gamma_1 + \gamma_2) \leq \nu(\gamma_1) + \nu(\gamma_2)$ (additivly written as $\pi_1(\mathrm{Ham}(M,\Omega))$ is abelian). We now want to take a look at the positive and negative parts of ν separately and set

$$\nu_+(\gamma) = \inf_F \int_0^1 \max_x F_t dt = \inf_F \max_{x,t} F(x,t)$$
$$\nu_-(\gamma) = \inf_F \int_0^1 -\min_x F_t dt = \inf_F (-\min_{x,t} F(x,t))$$

where the latter equalities have been show in an earlier lecture. Since they are defined as infima it holds that $\nu_+(\gamma) = \nu_-(-\gamma)$ and $\nu(\gamma) \geq \nu_+(\gamma) + \nu_-(\gamma)$. Recall that for S^2 with a normalized volume form Ω, such that the total volume is 1, we have $\mathrm{Ham}(S^2,\Omega) = \mathrm{Symp}_0(S^2,\Omega)$ and $\pi_1(\mathrm{Ham}(S^2)) \cong \mathbb{Z}_2$. The non-trivial element is generated by a rotation around the x_3-axis with Hamiltonian $F = \frac{1}{2}x_3$ (the normalization here essentially divides everything by 4π).Therefore $\max F = -\min F = \frac{1}{2}$ and thus $\nu_+(\gamma) \leq \frac{1}{2}$. Actually equality holds here (Theorem 9.1.A in [Pol12]). To show this is our goal now as it allows us to compute $\nu(\gamma) = 1$ as follows: Denote by $\{f_t\}$ the Hamiltonian loop of $F = \frac{1}{2}x_3$, then $\mathrm{length}\{f_t\} = \frac{1}{2} - (-\frac{1}{2}) = 1$, thus $\nu(\gamma) \leq 1$ where $\gamma \in \pi_1(\mathrm{Ham}(S^2,\Omega))$ is generated by $\{f_t\}$. Since $\gamma = -\gamma$ we have $\nu_-(\gamma) = \nu_+(-\gamma) = \frac{1}{2}$ and therefore $1 = \nu_+(\gamma) + \nu_-(\gamma) \leq \nu(\gamma) \leq 1$. Such Hamiltonian loops $\{f_t\}$ representing $\gamma \neq 0$ with $\mathrm{length}\{f_t\} = \nu(\gamma)$ are called *closed minimal geodesics*. As the starting point and ending point of a loop are equal these closed minimal geodesics aren't strictly minimal geodesics.

Symplectic fibrations over S^2

Definition. *Let F be a smooth manifold. A locally trivial fibration with fiber F is a map $p : P \to B$ between smooth manifolds and an open cover $\{U_\alpha\}_{\alpha \in A}$ with differmorphisms ϕ_α, such that for all $\alpha \in A$ the diagram*

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow[\cong]{\phi_\alpha} & U_\alpha \times F \\ & {}_{p}\searrow \quad \swarrow_{\mathrm{proj}_{U_\alpha}} & \\ & U_\alpha & \end{array}$$

commutes, where proj *denotes the projection. We call P the total space, B the base space and Φ_α a local trivialisation. For all $b \in B$ and $\alpha \in A$ we get a map*

$$\phi_{\alpha,b} = \phi_\alpha|_{p^{-1}(b)} \circ \mathrm{proj}_F : p^{-1}(b) \to F$$

As $p^{-1}(b) \cong F$ these maps can be viewed as elements of $\mathrm{Diff}(F)$*. We say the fibration is symplecic if the fiber F is a symplectic manifold (M,Ω) and the fibration has structure group* $\mathrm{Symp}(M,\Omega)$*, meaning all the transition functions*

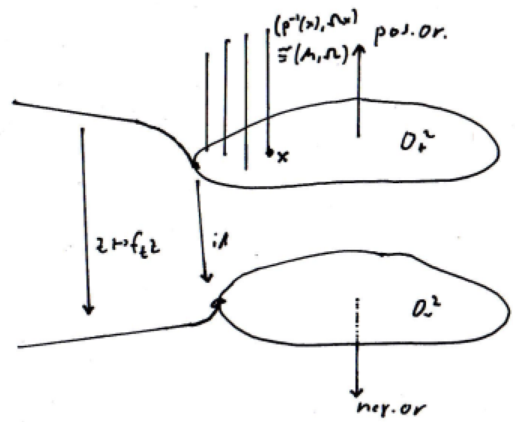
$$\phi_{\beta,b} \circ (\phi_{\alpha,b})^{-1} : F \to F$$

are symplectic whenever $b \in U_\alpha \cap U_\beta$.

Henceforth we consider all symplectic fibrations to be over S^2. Let (M,Ω) be a closed symplectic manifold with the additional assumption that $H^1(M,\mathbb{R}) = 0$ and therefore $\mathrm{Ham}(M,\omega) = \mathrm{Symp}_0(M,\Omega)$ (see remark 1.4.C in [Pol12]). Let $p : P \to S^2$ be a symplectic fibration with fiber (M,Ω) such that all fibers $p(x)^{-1}$ for $x \in S^2$ have a symplectic form Ω_x that varies smoothly with x such that all $(p^{-1}(x),\Omega_x)$ are symplectomorphic to (M,Ω). This can be achieved by giving each fiber the pullback form $\Omega_x = \phi^*_{\alpha,x}\Omega$ for $x \in U_\alpha$. Note that this is independent of the choice of trivialisation as the transition functions are symplectomorphic.
For a loop of Hamiltonian differmorphisms $\{f_t\}$ of (M,Ω) we now construct a symplectic fibration over S^2 as follows: Take two closed unit 2−disks D^2_+ and D^2_- with opposite orientation and define

$$P = M \times D^2_- \cup_\psi M \times D^2_+$$

with $\psi : M \times S^1 \to M \times S^1, (z,t) \mapsto (f_t z, t)$, where we identify $S^1 \cong \mathbb{R}/\mathbb{Z}$.



Note that this construction gives an orientation to S^2. If two Hamiltonian loops are homotopic the resulting fibrations are isomorphic. On the other hand if given a symplectic fibration over S^2 one can reconstruct the homotopy class γ giving rise to an isomorphic fibration as follows: First take two antipodal closed hemispheres. As these hemispheres are contractible the locally trivial fibration can be trivialized over the entire hemisphere. Comparing the borders now gives rise to a loop of symplectomorphisms that determine γ. We write $P(\gamma)$ for the fibration given by γ. Note that $P(0) = S^2 \times (M,\Omega)$, the trivial fibration.
For the case where the fiber is also S^2 and γ of the 1−turn rotation, a to $P(\gamma)$ isomorphic fibration can be constructed as follows: Recall that for a vector space V over a field $\mathbb{K}$ the projectivization $\mathbb{P}V$ is defined as the orbit space of $V \setminus \{0\}$ under the action of multiplication by the multiplicative group $\mathbb{K}\setminus\{0\}$. Identifying $S^2 \cong \mathbb{C}P^1 (:= \mathbb{P}(\mathbb{C}^2))$ we take two vector bundles: C, the trivial one $\mathbb{C} \times \mathbb{C}P^1$ and T, the tautological line bundle where the fibers are just the orbits within $\mathbb{C}^2$ adding 0 (a one-dimensional complex subspace). We now define $\mathbb{P}(T \oplus C)$ as the fibration over $\mathbb{C}P^1$ where we first take the Whitney sum $T \oplus C$ and then projectivize in each fiber. Note that in $T \oplus C$ each fiber is isomorphic to $\mathbb{C}^2$ so their projectivization is indeed isomorphic to $\mathbb{C}P^1 \cong S^2$.

Symplectic connections

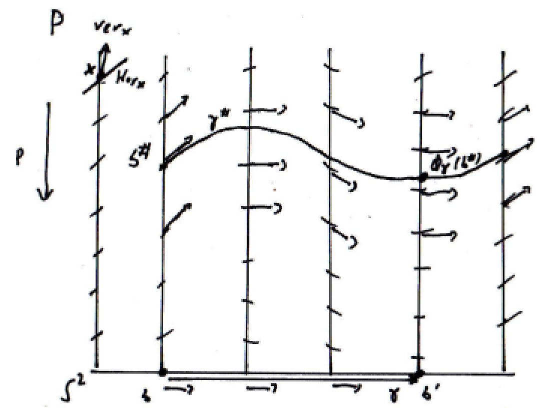
Definition. *Let $p : P \to B$ be a symplectic fibration with fiber (M,Ω). For $x \in P$ denote by*

$$\mathrm{Vert}_x = \ker dp(x) = Tp^{-1}(p(x))_x$$

the vertical tangent space. A connection σ on P is collection of horizontal subspaces Hor_x *such that*

$$TM_x = \mathrm{Vert}_x \oplus \mathrm{Hor}_x$$

Restricting $dp(x)$ to Hor_x now defines an isomorphism $dp(x)|_{\mathrm{Hor}_x} : \mathrm{Hor}_x \to TB_x$ allowing us to uniquely lift a vector field on B to P. Every smooth (for simplicity regular and simple) path γ in B from b to b' can now be lifted, given a fixed starting point $b^\sharp \in p^{-1}(b)$, by first locally extending and then lifting it's velocity vector field followed by taking the integral curve starting at $b^\sharp$. This defines diffeomorphisms $\Phi_\gamma : p^{-1}(b) \to p^{-1}(b')$, which only depend on the homotopy type of γ relative endpoints, called parallel transports and we say the connection is symplectic if these maps preserve the symplectic structure of the fibers, meaning $\Phi_\gamma^* \Omega_{b'} = \Omega_b$.



We now define the curvature ρ^σ of a connection σ as follows: Given $x \in S^2$ and $\xi, \eta \in TS^2_x$ we extend ξ and η locally and lift it to the connection to $\xi^\sharp$ and $\eta^\sharp$ and define

$$\rho^\sigma(\xi, \eta) = ([\xi^\sharp, \eta^\sharp])^{\mathrm{Vert}}$$

Here Vert denotes the projection onto the vertical components. This produces a vector field in the lie algebra of $\mathrm{Symp}(p^{-1}(x))$, therefore a Hamiltonian vector field as $H^1(M, \mathbb{R}) = 0$. We can therefore identify it with it's unique normalized Hamiltonian and therefore view $\rho^\sigma(\xi, \eta)$ as a function $p^{-1}(x) \to \mathbb{R}$.
Given an normalized area form τ on S^2, which we gave an orientation, we can use the fact that all $2-$forms are a multiple of τ to set

$$\rho^\sigma = L^\sigma \tau$$

where $L^\sigma : P \to \mathbb{R}$. Therefore ρ^σ is a function taking, for a given $x \in S^2$, two elements $\xi, \tau \in TS^2_x$ and a point of $p^{-1}(x)$ to $\mathbb{R}$.
For all $(x, z) \in P$ we can write the tangent space as

$$TP_{(x,z)} = T(p^{-1}(x))_z \oplus TS^2_x$$

Using this decomposition we can now define the *coupling form* of σ as the $2-$form on P

$$\delta^\sigma(v \oplus \xi, w \oplus \eta) = \Omega_x(v, w) - \rho(\xi, \eta)(z)$$

One can show that δ^σ is closed, thus let c be it's class in $H^2(P, \mathbb{R})$. Restricting c to a fiber $p^{-1}(x)$ gives the class of Ω_x (remember Ω_x defines a cohomology class as we assumed that $H^1(M, \mathbb{R}) = 0$).

Theorem (9.3.A in [Pol12], for a proof see [MS17])**.** *The class c is the unique cohomology class in $H^2(P, \mathbb{R})$ such that it restricts to $[\Omega_x]$ on the fibers and $c^{n+1} = 0$ where $2n = \dim M$.*

We now introduce the *weak coupling construction*. For $\varepsilon > 0$ small enough and $t \in [0, \varepsilon)$ there exists a smooth family of closed $2-$forms ω_t on P such that:

- $\omega_0 = p^*\tau$
- $[\omega_t] = tc + p^*[\tau]$ for all t
- ω_t restricts to Ω_x on the fibers

- ω_t is symplectic for $t > 0$

We now set $\varepsilon(P) = \sup \varepsilon$ where the supremum is taken over all such weak coupling constructions. Also note that $\varepsilon(P(0)) = +\infty$. We also define

$$\chi_+(P) = \sup_\sigma \frac{1}{\max_P L^\sigma}$$

Note that these constructions don't depend on our choice of τ.

Theorem (9.3.B of [Pol12])**.** *It holds that* $\varepsilon(P(\gamma)) \geq \chi_+(P(\gamma)) \geq \frac{1}{\nu_+(\gamma)}$.

In fact these inequalities are actually equalities, but these inequalities suffice for our purpose.

Proof. We write P for $P(\gamma)$ and first show that $\varepsilon(P) \geq \chi_+(P)$: Let σ be a symplectic connection on P and define

$$\omega_t = p^*\tau + t\delta^\sigma$$

where δ^σ is the coupling form of σ defined earlier. Evaluating at $(x, z) \in P$ yields

$$\omega_{t,(x,z)} = t\Omega_x \oplus (-tL^\sigma(x, z)\tau) = t\Omega_x \oplus ((1 - tL^\sigma(x, z))\tau)$$

It clearly holds by definition that $\omega_0 = p^*\tau$, $[\omega_t] = tc + p^*[\tau]$ and from the ladder equality one sees that it restricts to $t\Omega_x$ on the fibers. To show that it is indeed a weak coupling construciton we need to show tht it is symplectic. Since $t\Omega_x$ is non-degenerate we need to show that $(1 - tL^\sigma(x, z))\tau$ isn't and since it starts at τ for $t = 0$ we need $1 - tL^\sigma(x, y) > 0$ for all $(x, y) \in P$ with in turn means

$$\frac{1}{\max_P L^\sigma(x, z)} > t$$

By definition for all $\kappa > 0$ small enough there exists a connection σ such that

$$\frac{1}{\max_P L^\sigma(x, z)} > \chi_+(P) - \kappa$$

We have therefore a construction that works for $t \in [0, \chi_+(P) - \kappa)$, thus $\varepsilon(P) \geq \chi_+(P) - \kappa$ and since $\kappa > 0$ was arbitrarily small we've shown the existence of such a construction and that

$$\varepsilon(P) \geq \chi_+(P)$$

We now show that $\chi_+(P) \geq \frac{1}{\nu_+(P)}$: For any closed 2−form ω on P that restricts to Ω_x on the fibers the subspaces

$$\sigma_{(x,z)} = \{\xi \in TP_{(x,z)} \mid \iota_\xi\omega = 0 \text{ on } Tp^{-1}(x)_z\}$$

form a complement to $Tp^{-1}(x)_z$ as the fibers are either two-dimensinal and $Tp^{-1}(x)_z$ is thus Lagrangian or ω is degenerate and $\sigma_{(x,z)}$ which is, due to the non-degeneracy of Ω_x, clearly a at most two-dimensional subspace transversal to $Tp^{-1}(x)_z$, is non-trivial and even dimensional. Therefore $\sigma_{(x,z)}$ forms a symplectic connection. Now take a loop of Hamiltonian diffeomorphisms $\{f_t\}$ generated by a normalized Hamiltonian $F \in \mathcal{H}$ and take polar coordinates $(u, t) \in (0, 1] \times \mathbb{R}/\mathbb{Z}$ on D^2 (excluding the center) and define a smooth monotonely growing cutoff function $\phi : [0, 1] \to [0, 1]$ that is constant 0 near 0 and constant 1 near 1. Set

$$P = M \times D^2_- \cup_\psi M \times D^2_+$$

with $\psi : M \times S^1 \to M \times S^2, (z, t) \mapsto (f_t z, t)$ and define a closed 2−form ω on P as

$$\omega = \begin{cases} \Omega & \text{on } M \times D^2_+ \\ \Omega + d(\phi(u)H_t(z)) \wedge dt & \text{on } M \times D^2_- \end{cases}$$

This is well defined as $\psi^*\Omega = \Omega + dH_t \wedge dt$.
We are now interested in calculating the curvature ρ^σ of the connection σ belonging to ω under the construction above. Note that $\rho^\sigma \equiv 0$ on D^2_+ and close to the south pole, which is good for us as our polar coordinates are singular at the poles. To calculate ρ^σ we now need to lift the coordinate vector fields $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ for all $(x, z) \in M \times D^2_-$ to

the connection:
First for $\frac{\partial}{\partial u}$: It is of the form

$$\widetilde{\frac{\partial}{\partial u}} = \frac{\partial}{\partial u} + v$$

for $v \in Tp^{-1}(x)_z$. By definition of our connection we have

$$0 = \omega\left(\widetilde{\frac{\partial}{\partial u}}, w\right) = \omega(v, w) = \Omega(v, w)$$

for all $w \in Tp^{-1}(x)_z$ it follows from non-degeneracy of Ω on the fibers that $v = 0$ and thus

$$\widetilde{\frac{\partial}{\partial u}} = \frac{\partial}{\partial u}$$

For $\frac{\partial}{\partial t}$ we set similarly

$$\widetilde{\frac{\partial}{\partial t}} = \frac{\partial}{\partial t} + v$$

for $v \in Tp^{-1}(x)_z$ and get as before that for all $w \in Tp^{-1}(x)_z$

$$0 = \omega\left(\widetilde{\frac{\partial}{\partial t}}, w\right) = \omega\left(\frac{\partial}{\partial t} + v, w\right) = \Omega(v, w) - d(\phi(u)H_t)(w)$$

which means $\iota_v \Omega = d(\phi(u)H_t)$ and therefore

$$v = -\phi(u)\, \mathrm{sgrad}\, H_t$$

implying

$$\widetilde{\frac{\partial}{\partial t}} = \frac{\partial}{\partial t} - \phi(u)\, \mathrm{sgrad}\, H_t$$

We can now calculate

$$\rho^\sigma\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right) = \left[\frac{\partial}{\partial t} - \phi(u)\, \mathrm{sgrad}\, H_t, \frac{\partial}{\partial u}\right]^{\mathrm{Vert}} = \left(\phi'(u)\, \mathrm{sgrad}\, H_t - \phi(u)\, \mathrm{sgrad}\, H_t \frac{\partial}{\partial u}\right)^{\mathrm{Vert}} = \phi'(u)\, \mathrm{sgrad}\, H_t$$

as $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right]^{\mathrm{Vert}} = 0$. Identifying Hamiltonian vector fields with normalized Hamiltonian functions as above we get

$$\rho^\sigma\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right) = \phi'(u) H_t(z)$$

Now fix $\kappa > 0$ and define an area form τ on S^2 such that it is of the form $(1-\kappa)dt \wedge du$ on D^2_- (which has negative orientation) and extend it to D^2_+ such that D^2_+ has area κ. We get $\rho^\sigma = L^\sigma(u, t, z)\tau$, where

$$L^\sigma(u, t, z) = \begin{cases} 0 & \text{on } M \times D^2_+ \\ \frac{\phi'(u) H_t(z)}{1-\kappa} & \text{on } M \times D^2_- \end{cases}$$

If we now choose ϕ such that $\phi' \le 1 + \kappa$ and $\{f_t\}$ such that $\max_z H_t = \max_z F_t \le \nu_+(\gamma) + \kappa$ we get

$$\max_P L^\sigma \le \frac{1+\kappa}{1-\kappa}(\nu_+(\gamma) + \kappa)$$

implying

$$\chi_+(P) = \sup_\sigma \frac{1}{\max_P L^\sigma} \ge \frac{1-\kappa}{(1+\kappa)(\nu_+(\gamma)+\kappa)}$$

Since κ was arbitrarily small we finally get

$$\chi_+(P) \ge \frac{1}{\nu_+(\gamma)}$$

□

An application to length spectrum

The final ingredient to proof that $\nu_+(\gamma) = \frac{1}{2}$ for γ the homotopy class of the one turn rotation of S^2 is the following theorem which will be part of a future presentation (chapter 10 of [Pol12]):

Theorem (9.4.A in [Pol12])**.** $\varepsilon(P(\gamma)) \leq 2$

Proof of 9.1.A. We already know that $\nu_+(\gamma) \leq \frac{1}{2}$. By 9.3.B we have

$$2 \geq \varepsilon(P(\gamma)) \geq \chi_+(P(\gamma)) \geq \frac{1}{\nu_+(\gamma)} \geq 2$$

and therefore $\nu_+(\gamma) = \frac{1}{2}$. □

References

[Pol12] Leonid Polterovich. *The Geometry of the Group of Symplectic Diffeomorphism.* Birkhäuser Basel, 2012. ISBN: 978-3-0348-8299-6.

[MS17] Dusa McDuff and Dietmar Salamon. *Introduction to Symplectic Topology.* Oxford University Press, Mar. 2017. ISBN: 9780198794899.