Motivation

One of the main result of the talk is the proof of $\varepsilon(P(\gamma)) \leq 2$, which implies $\nu_+(\gamma) = \frac{1}{2}$ (as shown by Marius), thus completing the computation of the length spectrum of $\mathrm{Ham}(S^2)$.
This is an instance of a general problem that requires pseudo-holomorphic curves to be solved.

Quasi - Kähler structure

Def Let M be a smooth manifold. An almost complex structure on M is an automorphism $J : TM \to TM$ s.t. $J^2 = -\mathbb{1}$
An almost complex manifold is a pair (M, J)
Denote the space of almost complex structures on M by

$$\mathcal{J}(M) := \{ J \in C^\infty(M, \mathrm{End}(TM)) \mid J^2 = -\mathbb{1} \}$$

Def Let (M, ω) be a symplectic manifold. An almost complex structure J on M is compatible if

$$\begin{aligned} g : TM \times TM &\longrightarrow \mathbb{R} \\ (\xi, \eta) &\longmapsto \omega(\xi, J\eta) \end{aligned}$$

is a Riemannian metric on M.

Def The almost complex structure J is integrable if M^{2n} can be covered by coordinate charts $\phi : U \to \phi(U) \subset \mathbb{R}^{2n}$ s.t. $\forall q \in M$

$$d\phi(q) \circ J = J_0 \circ d\phi(q) : T_qM \longrightarrow \mathbb{R}^{2n}$$

where $J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$

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Def Let (M,ω) be a symplectic manifold and J a compatible
almost complex structure. The triple (M,ω,J) is a <u>quasi-Kähler manifold</u>
If J is integrable, (M,ω,J) is a <u>Kähler manifold</u>.

Quasi-Kähler manifold are relevant as all symplectic manifolds can be given a compatible almost complex structure. This follows from contractibility of $\mathcal{J}(M)$
However there are examples of non-kähler symplectic manifolds.

Def A map $\phi : (S^2, i) \longrightarrow (P, j)$ is a pseudo-holomorphic curve if $\phi_* \circ i = j \circ \phi_*$
or equivalently $\bar{\partial}\phi := \frac{1}{2}\left(\phi_* + j \circ \phi_* \circ i\right) = 0$

The deformation problem

Let (P,ω) be a closed symplectic manifold and ℓ a ray in $H^2(P,\mathbb{R})$
with origin at $[\omega]$.
How far can one deform ω with symplectic structures on P s.t $[\omega]$
moves on ℓ ?

Weak coupling construction

For $\varepsilon > 0$ sufficiently small, there exists a smooth family of closed 2-forms ω_t s.t. on P $(t \in [0,\varepsilon))$ s.t

- $\omega_0 = p^*\tau$
- $[\omega_t] = tc + p^*[\tau]$
- $\omega_t|_{fiber} = t\Omega_x$
- ω_t symplectic for all $t > 0$

We see that $p^*[\tau]$ is degenerate, but for $t > 0$ $tc + p^*[\tau]$ is
Denote $\varepsilon(P) = \sup \varepsilon$. We construct an example where $\varepsilon(P) < \infty$

Homological intermezzo

The cohomology groups $H^n(P)$ can be turned into rings by means of the cup product :

$$\cup : H^s(P) \times H^t(P) \longrightarrow H^{s+t}(P)$$
$$(\alpha, \beta) \longmapsto \alpha \cup \beta$$

This is a bilinear operation that respects the grading.

We also need Poincare duality : for a closed oriented manifold P

$$H^p(P) \cong H_{n-p}(P) \qquad \forall p \in \mathbb{N}_0$$

P is symplectic in our case so it is orientable.

The Poincare dual of the cup product is the intersection product

If $\alpha \in H^s(P)$, $\beta \in H^t(P)$ and the duals are $A \in H_{n-s}(P)$, $B \in H_{n-t}(P)$

$$\alpha \cup \beta \in H^{s+t}(P)$$
$$(A, B) \in H_{n-(s+t)}(P)$$

With the intersection product, homology on a manifold becomes a ring.

Assume now that P is connected : $H^0(P) \cong \mathbb{Z}$

We define $[P]$, the <u>fundamental class of N</u>, as the unique element of $H_n(P)$ that is sent to 1 under Poincare duality : $\Phi([P]) = 1$

If $N \subseteq M$ is a submanifold of M, the inclusion $i : N \hookrightarrow M$ induces a homorphism $i_* : H_n(N) \longrightarrow H_n(M)$.

Then $i_*([N]) \in H_n(M)$ is called the <u>class of N in M</u>

Let p, q s.t $p + q = \dim M =: m$ and

$$\left.\begin{array}{l} A \in H_p(M) \Rightarrow A^* \in H^q(M) \\ B \in H_q(M) \Rightarrow B^* \in H^p(M) \end{array}\right. \Rightarrow A^* \cup B^* \in H^m(M)$$

$$(A, B) = (A^* \cup B^*)^* \in H_0(M) \cong \mathbb{Z}$$

This is called <u>intersection index of A and B</u>.

If $\dim N = \frac{1}{2} \dim M$, $([N], [N])$ is called <u>intersection index of N</u>

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The intersection product counts the number of intersections (with orientation).

<u>Def</u> Let P^4 be a 4-dim. symplectic manifold and Σ a symplectic embedded 2-sphere. If $([\Sigma],[\Sigma]) = -1$, we call Σ an <u>exceptional shpere</u>

<u>Thm 10.1</u> Let $\Sigma \subset (P^4,\omega)$ be an exceptional sphere and ω_t a deformation of symplectic form $(t \in [0,1])$. Then $([\omega_t],\Sigma) > 0$

<u>Application to coupling</u>

We study the coupling deformation for the fibration $\mathbb{P}(T \oplus \mathbb{C}) \longrightarrow \mathbb{C}P^1$, where T and $\mathbb{C}$ are the tautological and trivial bundle respectively.
Let $[F]$ be the class of the fiber and Σ be the section corresponding to the rank 1 subbundle $0 \oplus \mathbb{C}$.
Since for $b \neq c$ $\;p^{-1}(b) \cap p^{-1}(c) = \emptyset$ and all these fibers are homologous, it follows $([F],[F]) = 0$. Also $([F],[\Sigma]) = 1$
It can be shown that $([\Sigma],[\Sigma]) = -1$

Let ω_t be a coupling deformation. By Moser's theorem, for t small ω_t is symplectomorphic to a positive symplectic form on $\mathbb{P}(T \oplus \mathbb{C})$ (a Kähler form)
WLOG we can assume ω_t is Kähler for t small.
It follows that Σ is symplectic and embedded. Together with $([\Sigma],[\Sigma]) = -1$ this implies that Σ is an exceptional sphere.
By theorem 10.1.A, $([\omega_t],[\Sigma]) > 0 \quad \forall t \in [0,1]$
Recall $[\omega_t] = p^*[\tau] + tc$, where $[\tau] \in H_2(S^2,\mathbb{Z})$ is a generator and c is the coupling class.
Using Poincare duality, we identify homology and cohomology and get the following relations:

$$(p^*[\tau],[F]) = 0 \qquad (p^*[\tau],[\Sigma]) = 1$$

$$(c,[F]) = 1 \qquad c^2 = 0$$

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The generators of $H_1(S^2,\mathbb{Z})$ are $[F]$ and $[\Sigma]$. These relations define $p^*[\tau]$ and $[\tau]$ uniquely.

$$p^*[\tau] = \alpha[F] + \beta[\Sigma] \qquad c = \gamma[F] + \delta[\Sigma]$$

$$0 = \alpha([F],[F]) + \beta([\Sigma],[F]) = \beta$$
$$1 = (p^*[\tau],[\Sigma]) = \alpha([F],[\Sigma]) = \alpha \quad \Rightarrow p^*[\tau] = [F]$$

$$1 = (c,[F]) = \gamma([F],[F]) + \delta([\Sigma],[F]) = \delta$$

$$0 = (c,c) = \gamma^2([F],[F]) + 2\gamma([F],[\Sigma]) + ([\Sigma],[\Sigma]) = 2\gamma - 1 \Leftrightarrow \gamma = \frac{1}{2}$$

Therefore $c = [\Sigma] + \frac{1}{2}[F]$

By using Thm 10.1A, it follows

$$0 < ([\omega_t],[\Sigma]) = ([F] + t[\Sigma] + \tfrac{t}{2}[F],[\Sigma])$$
$$= ([F],[\Sigma]) + t([\Sigma],[\Sigma]) + \tfrac{t}{2}([F],[\Sigma])$$
$$= 1 - t + \tfrac{t}{2}$$

$\Rightarrow t < 2$

This is true for all weak deformations, so $\varepsilon(P) < 2$. This completes the proof ■

<u>Proprieties of pseudo-holomorphic curves</u>

Let (P^{2n},ω) a symplectic manifold and $A \in H_2(P,\mathbb{Z})$ a primitive class ($\nexists\, B \in H_2(P,\mathbb{Z})$ and $k>1$ integer s.t $A = kB$). In particular $A \neq 0$.

Let $\mathcal{J} = \{ J : TP \to TP \mid J^2 = -\mathbb{1}$ and J is ω-compatible$\}$ and
$\mathcal{N} := \{ f \in C^\infty(S^2,P) \mid [f] = A \}$.

Then $\mathcal{X} := \{ (f,j) \in \mathcal{N} \times \mathcal{J} \mid \bar\partial_j f = 0 \}$. $\mathcal{X}$ is a smooth manifold and

Then $\mathcal{X} := \{(f,j) \in \mathcal{N} \times \mathcal{J} \mid \overline{\partial}_j f = 0\}$. $\mathcal{X}$ is a smooth manifold and the projection $\pi : \mathcal{X} \longrightarrow \mathcal{J}$ is a Fredholm operator:

$$\text{Index } \pi_* = \dim \ker \pi_* - \dim \operatorname{coker} \pi_* = 2(c_1(A) + n)$$

Let j_0, j_1 regular values of π. So $\pi^{-1}(j_0)$, $\pi^{-1}(j_1)$ are smooth submanifold.
If γ is a pact from j_0 to j_1, $\pi^{-1}(\gamma)$ is a smooth submanifold of dimension $\text{Index}(\pi) + 1$ and $\partial \pi^{-1}(\gamma) = \pi^{-1}(j_0) \cup \pi^{-1}(j_1)$.

Key point: let $PSL(2,\mathbb{C})$ be the group of conformal transformations of (S^2, i)
If $h \in PSL(2,\mathbb{C})$ and $(f,j) \in \pi^{-1}(j)$, also $(f \circ h, j) \in \pi^{-1}(j)$
$\pi^{-1}(\gamma)$ can't be compact as it admits the action of the non-compact group $PSL(\mathbb{Z})$.

Thm (Gromov) Either $\pi^{-1}(\gamma)/_{PSL(2,\mathbb{C})}$ is compact or there exists a family $(f_k, j_k) \in \pi^{-1}(\gamma)$ s.t. $j_k \longrightarrow j_\infty$ and f_k converges to a j_∞-holomorphic cusp curve in the class A.
Write $A = \sum_{i=1}^{d} A_i$ and $\phi_k : S^2 \longrightarrow P$ are j-holomorphic curves in A_i.
The union $\bigcup_{i=1}^{d} \phi_i(A_i)$ is the image of the cusp curve, usually assumed to be connected.
Imagine now the following:

- The set $\mathcal{J}' \subseteq \mathcal{J}$ of all j that generate cusp curves has codimension at least 2
- $j \in \mathcal{J} \setminus \mathcal{J}'$ is a regular value and $\pi^{-1}(j_0)/_{PSL(2,\mathbb{C})}$ does not bound a compact manifold.

Then $\forall j_1 \in \mathcal{J} \setminus \mathcal{J}'$, $\pi^{-1}(j_1)/_{PSL(2,\mathbb{C})} \neq \emptyset$.
Indeed, suppose the contrary and join j_0 and j_1 by a pact γ. Then

$$\partial\left(\pi^{-1}(\gamma)/_{PSL(2,\mathbb{C})}\right) = \pi^{-1}(j_0)/_{PSL(2,\mathbb{C})}$$

But from Gromov's theorem, $\pi^{-1}(\gamma)/_{PSL(2,\mathbb{C})}$ is compact. This contradicts the second assumption.

Persistence of exceptional spheres

We sketch a proof for 10.1A

Let (P^4, ω) be a symplectic manifold and $\Sigma \subset (P^4, \omega)$ be an exceptional sphere with $A = [\Sigma]$. Let ω_t $(t \in [0,1])$ be a deformation of $\omega = \omega_0$

Step 1 : choose a j_0 that is ω-compatible and extend to a family j_t that is ω_t-compatible

Step 2 : It holds $\text{Index}\ \pi = 2(c_1(A) + n) = 2(1+2) = 6$

Since $\dim_{\mathbb{R}} PSL(2,\mathbb{C}) = 6$, it follows that if j_0 is regular

$$\dim_{\mathbb{R}} \left(\pi^{-1}(j_0)/PSL(2,\mathbb{C})\right) = 0$$

Step 3 In dimension 4, two different germs of j-holomorphic curves always intersect with positive index at common points.

Assume Σ' is another curve in A. Then

$$([\Sigma],[\Sigma']) = ([\Sigma],[\Sigma]) = -1 \ \lightning$$

It follows that there is a unique j-holomorphic curve in A.

$\pi^{-1}(j_0)/PSL(2,\mathbb{C})$ contains just one point.

Step 4 Choose j_0, j_t regular. We claim that no cusp curve appear generically

Set

$$A = \sum_{i=1}^{d} A_i \Rightarrow c_1(A) = \sum_{i=1}^{d} c_1(A_i) = 1$$

Thus at least one Chern class is non-positive. WLOG let it be $c_1(A)$

$$\dim \pi_{A_1}^{-1}(\gamma)/PSL(2,\mathbb{C}) = 2(c_1(A_1) + 2) - 6 + 1 \leq -1 \ \lightning$$

Generically no bobbling of happens. Thus A is represented by a j_t-hol.

By compactness $([\omega_t],[\Sigma]) > 0 \ \forall t \in [0,1]$.