Midterm exam

1. Consider the function

$$f : [-1, 1] \to \mathbb{R} : \quad f(x) = x^2 \left(\frac{1}{4} x^2 - \frac{1}{3} x - 1 \right).$$

For which $x \in [-1, 1]$ is $f(x)$ minimized and for which x is it maximized? Sketch the function on the interval $[-1, 1]$, paying special attention to the local extrema.

2. (a) What is

$$\frac{\mathrm{d}}{\mathrm{d}t} F(x(t), y(t)) \quad \text{at } t = 0$$

if $x(0) = 2$, $y(0) = 5$, $x_t(0) = -3$, $y_t(0) = 7$, $F_x(2, 5) = 8$ and $F_y(2, 5) = 2$?
Note that here we are using the notation $F_x = \frac{\partial F}{\partial x}$ which is more convenient when we have to evaluate the derivatives at a specific point.

(b) Denote $g(t) = F(x(t), y(t))$. Let $F(2, 5) = 2$. Determine the tangent line to the function g at $t = 0$.

3. (a) Parametrise the curve $x^{2/3} + y^{2/3} = 1$ from $\left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$ to $(1, 0)$.

(b) Compute the length of the curve $x^{2/3} + y^{2/3} = 1$ from $\left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$ to $(1, 0)$.

4. Find the solution $y(x)$ of the differential equation

$$\frac{dy}{dx} + y \tan x = 0$$

satisfying the initial condition $y\left(\frac{\pi}{3}\right) = 4$.