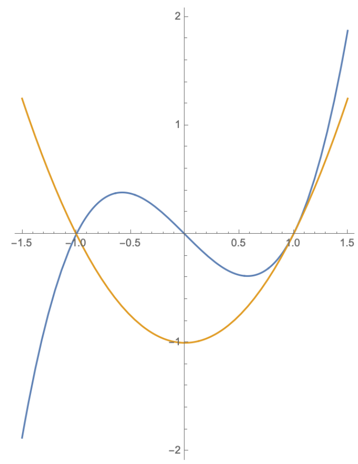
Solutions Midterm exam

1. The graph of the functions is



(a) We have to find x_i, $i = 1, 2$, such that $f(x_i) = g(x_i)$.

$$\begin{aligned} f(x) - g(x) &= x^3 - x - (x^2 - 1) \\ &= x(x^2 - 1) - (x^2 - 1) \\ &= (x - 1)\underbrace{(x^2 - 1)}_{(x-1)(x+1)} \\ &= (x - 1)^2 (x + 1) \\ &\overset{!}{=} 0 \end{aligned}$$

This is satisfied for $x_1 = -1$ and $x_2 = 1$.
We have $f(x_1) = f(-1) = 0$, $g(x_1) = g(-1) = 0$, $f(x_2) = f(1) = 0$ and $g(x_2) = g(1) = 0$.

(b) We now have to determine if $f(x) \geqslant g(x)$ or $g(x) \geqslant f(x)$ on the interval $[-1, 1]$. Since the functions f and g are continuous, we just need to check it in one point. Since $f(0) = 0$ and $g(0) = -1$, we see that $f(x) \geqslant g(x)$ for $-1 \leqslant x \leqslant 1$.

Another argument uses the derivative. Indeed $f'(x) = 3x^2 - 1$ and $g'(x) = 2x$. Hence

$$f'(-1) = 2 > -2 = g'(-1)$$

and this means that $f(x) \geq g(x)$ for $-1 \leqslant x \leqslant 1$.
The area enclosed by the curves is given by the following integral.

$$\begin{aligned}
\int_{-1}^{1} f(x) - g(x) dx &= \int_{-1}^{1} x^3 - x - x^2 + 1 dx \\
&= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 - \frac{1}{3}x^3 + x\right]_{-1}^{1} \\
&= \frac{1}{4} - \frac{1}{2} - \frac{1}{3} + 1 - \left(\frac{1}{4} - \frac{1}{2} + \frac{1}{3} - 1\right) \\
&= -\frac{2}{3} + 2 = \frac{4}{3}
\end{aligned}$$

2. We know that

$$e^{ix} = \cos(x) + i\sin(x)\,.$$

Comparing the real part of

$$\left(e^{ix}\right)^3 = e^{i\,3x} = \cos(3x) + i\sin(3x)$$

and

$$\begin{aligned}
\left(e^{ix}\right)^3 &= \big(\cos(x) + i\sin(x)\big)^3 \\
&= \cos^3(x) + 3i\cos^2(x)\sin(x) - 3\cos(x)\sin^2(x) - i\sin^3(x) \\
&= \cos^3(x) - 3\cos(x)\sin^2(x) + i\big(3\cos^2(x)\sin(x) - \sin^3(x)\big)
\end{aligned}$$

we get

$$\begin{aligned}
\cos(3x) &= \cos^3(x) - 3\cos(x)\sin^2(x) \\
&= \cos^3(x) - 3\cos(x)\big(1 - \cos^2(x)\big) \\
&= 4\cos^3(x) - 3\cos(x) \\
&= \cos(x)\big(4\cos^2(x) - 3\big)\,.
\end{aligned}$$

Remark:

$$\begin{aligned}
\sin(3x) &= 3\cos^2(x)\sin(x) - \sin^3(x) \\
&= 3\sin(x)\big(1 - \sin^2(x)\big) - \sin^3(x) \\
&= \sin(x)\big(3 - 4\sin^2(x)\big)\,.
\end{aligned}$$

3. By separating variables, we get

$$\begin{aligned}
&y' + xy + Cx = 0 \\
\Longleftrightarrow & y' + x(y+C) = 0 \\
\Longleftrightarrow & y' = -x(y+C) \\
\Longleftrightarrow & -y' = x(y+C) \\
\Longleftrightarrow & -\frac{\mathrm{d}y}{\mathrm{d}x} = x(y+C) \\
\Longleftrightarrow & -\frac{\mathrm{d}y}{y+C} = x\mathrm{d}x \\
\Longleftrightarrow & -\int \frac{\mathrm{d}y}{y+C} = \int x\mathrm{d}x \\
\Longleftrightarrow & -\ln(|y+C|) = \frac{1}{2}x^2 + C_1 \ \text{ for some } C_1 \in \mathbb{R} \\
\Longleftrightarrow & \ln(|y+C|) = -\frac{1}{2}x^2 - C_1 \ \text{ for some } C_1 \in \mathbb{R} \\
\Longleftrightarrow & |y+C| = C_2 e^{-\frac{1}{2}x^2} \ \text{ for some } C_2 \in \mathbb{R}^+ \\
\Longleftrightarrow & y(x) = -C + C_2 e^{-\frac{1}{2}x^2} \ \text{ for some } C_2 \in \mathbb{R}.
\end{aligned}$$

The two conditions

$$0 = y(0) = -C + C_2 e^{-\frac{1}{2}0^2} = -C + C_2 \Longleftrightarrow C = C_2,$$

$$0 = y\left(\sqrt{2}\right) + 1 - \frac{1}{e} = -C + C_2 e^{-\frac{1}{2}\sqrt{2}^2} + 1 - \frac{1}{e} = 1 - C + \frac{C_2 - 1}{e} = 1 - C + \frac{C-1}{e},$$

imply that

$$C = C_2 = 1.$$

The solution to the differential equation

$$y' + xy + x = 0$$

with the searched constant $C = 1$ and $y(0) = y\left(\sqrt{2}\right) + 1 - \frac{1}{e} = 0$, is therefore given by

$$y(x) = e^{-\frac{1}{2}x^2} - 1.$$

4. We first determine the general solution of the homogeneous equation

$$y'' - 4y' + 4y = 0\,.$$

The zeros of the characteristic polynomial $\lambda^2 - 4\lambda + 4$ are

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-16}}{2} = 2\,.$$

Hence the solution is

$$y_h(x) = C_1 x e^{2x} + C_2 e^{2x}\,.$$

For the particular solution of the inhomogeneous equation we guess

$$y_p(x) = A\cos(x) + B\sin(x)$$

Then

$$\begin{aligned} y_p'(x) &= -A\sin(x) + B\cos(x) \\ y_p''(x) &= -A\cos(x) - B\sin(x) \end{aligned}$$

and

$$\begin{aligned} y'' - 4y' + 4y &= -A\cos(x) - B\sin(x) \\ &\quad - 4\big(-A\sin(x) + B\cos(x)\big) \\ &\quad + 4\big(A\cos(x) + B\sin(x)\big) \\ &= (3A - 4B)\cos(x) + (4A + 3B)\sin(x) \\ &\overset{!}{=} \sin(x)\,. \end{aligned}$$

Hence we solve

$$\begin{aligned} 3A - 4B &= 0 \\ 4A + 3B &= 1 \end{aligned}$$

and get

$$A = \frac{4}{25} \text{ and } B = \frac{3}{25}\,.$$

The particular solution of the inhomogeneous equation is

$$y_p(x) = \frac{4}{25}\cos(x) + \frac{3}{25}\sin(x)$$

and the general solution is

$$y(x) = C_1 x e^{2x} + C_2 e^{2x} + \frac{4}{25}\cos(x) + \frac{3}{25}\sin(x)$$

The initial conditions determine the constants C_1 and C_2. First we compute

$$y'(x) = C_1(1 + 2x)e^{2x} + 2C_2 e^{2x} - \frac{4}{25}\sin(x) + \frac{3}{25}\cos(x)\,.$$

We solve

$$\begin{aligned} y(0) &= C_2 + \frac{4}{25} \overset{!}{=} \frac{1}{5} \\ y'(0) &= C_1 + 2C_2 + \frac{3}{25} \overset{!}{=} 1 \end{aligned}$$

and get $C_2 = \frac{1}{25}$, $C_1 = \frac{4}{5}$. The solution is

$$y(x) = \frac{4}{5} x e^{2x} + \frac{1}{25} e^{2x} + \frac{4}{25}\cos(x) + \frac{3}{25}\sin(x)$$