1. **Area enclosed by two curves**

   Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that

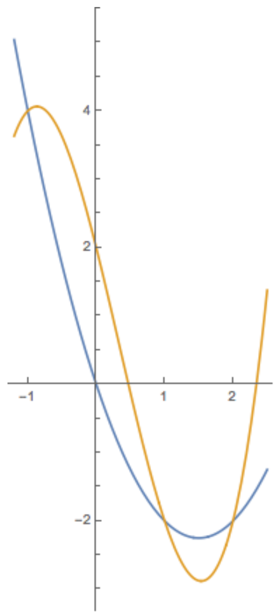
   $$f(x) = x^3 - x^2 - 4x + 2\,, \quad \text{and} \quad g(x) = x^2 - 3x\,.$$

   (a) Determine the points $x_1 < x_2 < x_3 \in \mathbb{R}$ in which the graphs of f and g intersect.

   (b) Compute the area that is enclosed by the graphs of f and g between x_1 and x_3.

   Hint: It may help to multiply $(x+1)(x-1)(x-2) =: h(x)$.

   Solution: The graphs of the functions are



   (a) We have to find x_i, $i = 1, 2, 3$, such that $f(x_i) = g(x_i)$.

   $$\begin{aligned} f(x) - g(x) &= (x^3 - x^2 - 4x + 2) - (x^2 - 3x) \\ &= x^3 - 2x^2 - x + 2 \\ &= (x+1)(x-1)(x-2) = h(x) \stackrel{!}{=} 0 \end{aligned}$$

   We get $x_1 = -1$, $x_2 = 1$, $x_3 = 2$.

(b) Since $f \geq g$ on the interval $[-1, 1]$ and $g \geq f$ on $[1, 2]$, the area is

$$\begin{aligned}
A &= \int_{-1}^{1} f(x) - g(x)\,dx + \int_{1}^{2} g(x) - f(x)\,dx \\
&= \int_{-1}^{1} x^3 - 2x^2 - x + 2\,dx + \int_{1}^{2} -x^3 + 2x^2 + x - 2\,dx \\
&= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-1}^{1} + \left[-\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 - 2x\right]_{1}^{2} \\
&= \frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2 - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} + 2 - \frac{16}{4} + \frac{16}{3} + \frac{4}{2} - 4 + \frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2 \\
&= -\frac{15}{4} + \frac{10}{3} + \frac{3}{2} + 2 \\
&= \frac{-45 + 40 + 18 + 24}{12} = \frac{37}{12}\,.
\end{aligned}$$

2. **Extrema**

Consider the function

$$f : [-1, 1] \to \mathbb{R} : \quad f(x) = x^2\left(\frac{1}{4}\,x^2 + \frac{1}{3}\,x - 1\right).$$

For which $x \in [-1, 1]$ is $f(x)$ minimized and for which x is it maximized?

Solution:

$$f'(x) = x^3 + x^2 - 2x = x(x-1)(x+2)$$

so the stationary points in the domain are at $x = 1$ and $x = 0$. We also need to check the boundary point $x = -1$. We have $f(-1) = -13/12$, $f(0) = 0$ and $f(1) = -5/12$. Therefore the minimum is attained at $x = -1$ and the maximum is attained at $x = 0$.

3. **Complex numbers**

Find all the solutions of the equation

$$z^5 = 4(1 + i)\,.$$

Solution: We compute

$$\begin{aligned}
z^5 = 4(1 + i) &= (\sqrt{2})^5\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) \\
&= \left(\sqrt{2}\right)^5 \exp\left(i\frac{\pi}{4}\right)
\end{aligned}$$

With the equations

$$\begin{aligned} r^5 &= \left(\sqrt{2}\right)^5 \\ 5\varphi &= \frac{\pi}{4} + k \cdot 2\pi\,, \quad k \in \mathbb{Z}\,, \end{aligned}$$

we get

$$\begin{aligned} r &= \sqrt{2} \\ \varphi &= \frac{\pi}{20} + k \cdot \frac{2\pi}{5}\,, \quad k \in \mathbb{Z}\,. \end{aligned}$$

There are five different solutions:

$$\begin{aligned} z_0 &\overset{k=0}{=} \sqrt{2}\,\exp\left(i\,\frac{\pi}{20}\right) \\ z_1 &\overset{k=1}{=} \sqrt{2}\,\exp\left(i\,\frac{9\pi}{20}\right) \\ z_2 &\overset{k=2}{=} \sqrt{2}\,\exp\left(i\,\frac{17\pi}{20}\right) \\ z_3 &\overset{k=3}{=} \sqrt{2}\,\exp\left(i\,\frac{25\pi}{20}\right) \\ z_4 &\overset{k=4}{=} \sqrt{2}\,\exp\left(i\,\frac{33\pi}{20}\right) \end{aligned}$$

4. **First order differential equation**

Find the solution $x(t)$ of the differential equation

$$x' = 2x\,te^t$$

subject to the condition that $x = 1$ when $t = 0$.

Solution: We separate the variables

$$\begin{aligned} \frac{dx}{dt} &= 2x\,te^t \\ \int \frac{dx}{x} &= \int 2te^t dt \\ \ln|x| &= 2te^t - 2\int e^t\,dt = 2\big(t-1\big)e^t + C \\ x(t) &= C\exp\big(2(t-1)e^t\big) \end{aligned}$$

The condition $x(0) = 1$ yields

$$\begin{aligned} x(0) &= C\exp\big(-2 \cdot e^0\big) = Ce^{-2} \overset{!}{=} 1 \\ C &= e^2 \\ x(t) &= \exp\big(2(t-1)e^t + 2\big) \end{aligned}$$