Midterm exam

1. Consider the function

$$f : [-1, 1] \to \mathbb{R} : \quad f(x) = x^2 \left(\frac{1}{4}\, x^2 - \frac{1}{3}\, x - 1 \right).$$

For which $x \in [-1, 1]$ is $f\,(x)$ minimized and for which x is it maximized? Sketch the function on the interval $[-1, 1]$, paying special attention to the local extrema.

Solution:

$$f'\,(x) = x^3 - x^2 - 2x = x\,(x + 1)\,(x - 2)$$

so the stationary points in the domain are at $x = -1$ and $x = 0$. We also need to check the boundary point $x = 1$. We have $f\,(-1) = -5/12$, $f\,(0) = 0$ and $f\,(1) = -13/12$. Therefore the minimum is attained at $x = 1$ and the maximum is attained at $x = 0$.

2. (a) What is

$$\frac{\mathrm{d}}{\mathrm{d}t} F(x(t), y(t)) \quad \text{at } t = 0$$

if $x(0) = 2\,,\ y(0) = 5\,,\ x_t(0) = -3\,,\ y_t(0) = 7\,,\ F_x(2, 5) = 8$ and $F_y(2, 5) = 2$?
Note that here we are using the notation $F_x = \frac{\partial F}{\partial x}$ which is more convenient when we have to evaluate the derivatives at a specific point.

(b) Denote $g(t) = F(x(t), y(t))$. Let $F(2, 5) = 2$. Determine the tangent line to the function g at $t = 0$.

Solution. $F_t(0) = 8 \cdot (-3) + 2 \cdot 7 = -10$.
Since $g(0) = F(2, 5) = 2$, the tangent at $t = 0$ is $y = -10x + 2$.

3. (a) Parametrise the curve $x^{2/3} + y^{2/3} = 1$ from $\left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$ to $(1, 0)$.

(b) Compute the length of the curve $x^{2/3} + y^{2/3} = 1$ from $\left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$ to $(1, 0)$.

Solution. One possible parametrisation is $\vec{r}(t) = (\cos^3 t, \sin^3 t)$ with $t \in \left[\frac{\pi}{3}, 0\right]$. (How did we get the endpoints? We solved the equations $\cos t_0 = \frac{1}{2}$ and $\cos t_1 = 1$, and checked that the y-coordinate is correct.)

Let's compute the speed:

$$\begin{aligned}\dot{\vec{r}}(t) &= \left(-3\cos^2 t\sin t, 3\sin^2 t\cos t\right), \\ |\dot{\vec{r}}(t)| &= \sqrt{9\cos^4 t\sin^2 t + 9\sin^2 t\cos^2 t} &= \sqrt{9\cos^2 t\sin^2 t} \\ &= 3\cos t\sin t\end{aligned}$$

Integrating,

$$\begin{aligned}L &= \int_{\frac{\pi}{3}}^{0} 3\cos t\sin t\,\mathrm{d}t \\ &= \frac{3}{2}\left[-\cos^2(t)\right]_{\frac{\pi}{3}}^{0} \\ &= \frac{3}{2}\left(-1+\frac{1}{4}\right) = -\frac{9}{8}.\end{aligned}$$

The negative result just says that we were integrating in the wrong direction, so the length is actually $\frac{9}{8}$.

Alternatively, we can use the formula $2\cos t\sin t = \sin(2t)$ to get

$$\begin{aligned}L &= \int_{\frac{\pi}{3}}^{0} \frac{3}{2}\sin(2t)\,\mathrm{d}t \\ &= \frac{3}{4}\left[-\cos(2t)\right]_{\frac{\pi}{3}}^{0} \\ &= \frac{3}{4}\left(-1-\frac{1}{2}\right) = -\frac{9}{8}.\end{aligned}$$

4. Find the solution $y(x)$ of the differential equation

$$\frac{dy}{dx} + y\tan x = 0$$

satisfying the initial condition $y\left(\frac{\pi}{3}\right) = 4$.

Solution. Separating variables, we get $\frac{dy}{y} = -\tan x dx$.
Integration gives the general solution $\ln y = \ln(\cos x) + c$, i.e. (after exponentiating and setting $e^c = A$),

$$y = A\cos x.$$

Introducing the initial condition, $y\left(\frac{\pi}{3}\right) = 4 = A\cdot\frac{1}{2}$, we fix the constant $A = 8$. The particular solution is therefore

$$y = 8\cos x.$$