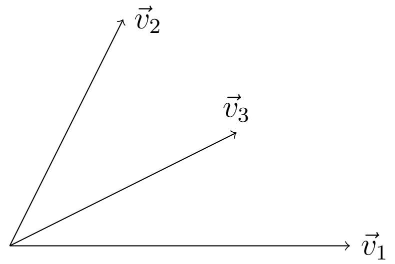
LINEAR (IN)DEPENDENCE, MATRICES AND DETERMINANT

1. For what values of parameter t is the following matrix invertible? If so, determine its inverse.

$$\begin{pmatrix} 1 & 0 & t \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

2. (a) Let $\vec{v}_1 = (3, 0)$, $\vec{v}_2 = (1, 2)$, $\vec{v}_3 = (2, 1)$ be vectors in $\mathbb{R}^2$.
Are $\vec{v}_1$, $\vec{v}_2$ linearly dependent? Justify your answer algebraically.
What about $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$?

   (b) Look at the drawing of the three vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ in the xy-plane. Try to answer 1.(a) geometrically, with no reference to the values of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.



3. (a) Are the following vectors linearly dependent or linearly independent?
   - $(1, 1, 1)$ and $(0, 1, -2)$,
   - $(1, 1, 1)$, $(1, 1, 0)$ and $(0, 0, -1)$,
   - $(1, 1, 1)$, $(1, 1, 0)$ and $(1, 0, -1)$.

   (b) For which values of $t \in \mathbb{R}$ are the vectors

$$(1, 1, 1),\ (1, 1, 0) \text{ and } (t, 0, -1)$$

   linearly dependent?

4. An $n \times n$ matrix is called *nilpotent* if $A^m = 0$ for some positive integer m.

   (a) Find an example of a nilpotent 3×3 matrix.

   (b) Consider a non zero nilpotent matrix A, and denote by m the smallest integer for which the nilpotency condition is satisfied. Let $\vec{x} \in \mathbb{R}^n$ be a vector such that $A^{m-1}\vec{x} \neq 0$. Show that the vectors $\vec{x}, A^1\vec{x}, A^2\vec{x}, \cdots, A^{m-1}\vec{x}$ are all linearly independent.

5. If possible, compute the following matrix products:

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$,

(c) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$,

(e) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

6. (a) By setting up an appropriate system of linear equations, find all vectors $\vec{x} \in \mathbb{R}^2$ such that $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

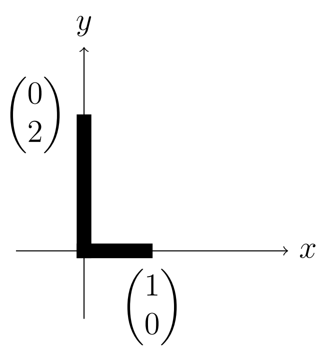
(b) Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a 2×2-matrix and $ad - bc \neq 0$. Derive a formula for the inverse of A by using the standard procedure for inverting matrices. Consider the cases $a \neq 0$ and $a = 0$ separately.

(c) Use the formula in (b) to compute $\vec{x}$ in (a).

7. The goal of this exercise is to give a geometric interpretation of the linear transformations defined by the matrices

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Start by showing the effect of these transformations on the letter L :



In each case, decide whether the transformation is invertible. Find the inverse if it exists, and interpret it geometrically.

8. Compute the determinants of the following matrices.

$$(a)\ A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad (b)\ B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 2 & 2 \\ 3 & 3 & 4 & 4 \end{pmatrix}$$

9. Compute the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \det \left[\begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{pmatrix} \right].$$

10. In the basis $\mathcal{B} = \left\{ \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) \right\}$, find the matrix M that represents the linear transformation $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined in the standard basis by

$$f_1(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x}.$$