Mathematics

Problem Set 10

DETERMINANTS, EIGENVALUES, EIGENVECTORS

1. For which values of $k \in \mathbb{R}$ is 5 an eigenvalue of the matrix

$$A_k = \begin{pmatrix} -1 & k \\ 4 & 3 \end{pmatrix}?$$

2. (a) Verify that the characteristic polynomial of a 2×2-matrix A is

$$p_A(\lambda) \ = \ \lambda^2 - \mathrm{tr}(A)\lambda + \det(A).$$

   (b) Let λ_1 and λ_2 be eigenvalues of A. Write down the relation between these eigenvalues, $\mathrm{tr}(A)$ and $\det(A)$.

3. Consider the linear mapping $f_2 : \mathbb{R}^3 \to \mathbb{R}^3$ defined in the standard basis by

$$f_2(\vec{x}) = A\vec{x}, \quad \text{where } A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

   (a) Compute the eigenvalues of the matrix A. For each eigenvalue, determine a corresponding eigenvector.

   (b) Find a diagonal matrix D and a corresponding transformation matrix S such that

$$S^{-1}AS = D\,.$$

4. Consider the linear mapping $f_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined in the standard basis by

$$f_3(\vec{x}) = A\vec{x}, \quad \text{where } A = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}, \quad \vartheta \in [0, 2\pi).$$

   (a) Describe the effects of f_3 on a generic vector $\vec{v} \in \mathbb{R}^2$.

   (b) Compute all eigenvalues of A. For each eigenvalue, find a corresponding eigenvector.

   (c) Find a matrix B that represents f_3 in the basis of eigenvectors.

5. (a) Transform the following system of linear equations to be in upper triangular form, where $\alpha \in \mathbb{R}$.

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \\ 2x & + & 3y & + & az & = & 3 \\ x & + & ay & + & 3z & = & 2. \end{array}$$

(b) Solve this system depending on the parameter α. In other words, determine how the set of solutions changes with α, and write down the corresponding solutions.

(c) Compute the determinant of the system. For which α is it zero? Explain the geometry behind the action of f_3 when A has vanishing determinant.

6. A 3×3-matrix B is known to have the eigenvalues $0, 1, 2$. This information suffices to determine which of the following? Give the answers wherever possible, or otherwise a justification.

   (a) the rank of B
   (b) the determinant of $B^\intercal B$
   (c) the eigenvalues of $B^\intercal B$
   (d) the eigenvalues of $(B^2 + I)^{-1}$
   (e) the trace of $(B^2 + I)^{-1}$

7. Determine the functions $x(t)$ and $y(t)$ that solve the system of linear differential equations

$$\dot{x} = 5x + 4y,$$
$$\dot{y} = 3x - 6y,$$

   subject to the initial conditions $x(0) = 13$ and $y(0) = 0$.

8. Determine the functions $x(t)$ and $y(t)$ that satisfy the system of linear differential equations

$$\dot{x} = -3x + 2y,$$
$$\dot{y} = -8x + 5y,$$

   subject to the initial conditions $x(0) = 0$ and $y(0) = 1$.

9. (Facultative: only to practive derivatives)

   Compute the gradient and set up the Hessian matrix of the real-valued function $f(x, y, z) = 3x^2y^2 + 5z^2$.

10. (Facultative: only to practive derivatives)

   A unit disk D including the boundary is heated so that the temperature at any point $(x, y) \in D$ is governed by the equation

$$T(x, y) \;=\; x^2 + 2y^2 - x.$$

   Find the temperatures at the hottest and coldest points on the disk.

11. (Facultative: only to practive derivatives and min max, but we don't expect you to compute min, max of multivariable functions)

    Consider at the function $f(x, y) = x^2y^2$ at $(0, 0)$. Determine whether the function has an extremum or not at the origin by imagining what the surface looks like. (You can try using the standard calculus methods, but you will notice that the Hessian matrix of f at $(0, 0)$ is the zero matrix.)