MATHEMATICS

Problem Set 5

COMPLEX NUMBERS

1. Write the following expressions in the form $x + iy$, with $x, y \in \mathbb{R}$.

   (a) $(3+5i)^2$ (b) $(-7+2i)(5-3i)$ (c) $|2+i|$

   (d) $\dfrac{i-1}{1+i}$ (e) $\dfrac{1-5i}{3i-1}$ (f) $\dfrac{1-i}{1+i}+2-i$

2. Write the expressions $-5+i$ and $2-3i$ in polar form $re^{i\varphi}$.

3. Find all solutions of the following equations:

   (a) $z^2 + 4z + 12 - 6i = 0$,
   (b) $z^6 = -4\sqrt{3} - 4i$,
   (c) $z^5 = (1+i)(-1+\sqrt{3}\,i)(\sqrt{3}-i)$.

4. Sketch the set of all $z \in \mathbb{C}$ satisfying the following conditions:

   (a) $\operatorname{Re} z > 2$, (b) $|z| < 3$, (c) $|z-1| < |z+1|$.

5. For a complex number $z = re^{i\varphi}$, the angle φ is the *argument* of z, and we write $\varphi = \arg z$. This representation is unique if $\varphi \in (-\pi, \pi]$, in which case φ is referred to as the *principal value* of $\arg z$. We get the same point if we replace φ by $\varphi + 2n\pi$, $n \in \mathbb{Z}$.

   (a) Let $r, s, \vartheta, \varphi \in \mathbb{R}$ with $r, s > 0$, and set

   $$z = r(\cos\vartheta + i\sin\vartheta), \quad w = s(\cos\varphi + i\sin\varphi).$$

   Compute the product $z \cdot w$ and use the formulae for $\sin(\vartheta+\varphi), \cos(\vartheta+\varphi)$ to deduce that

   $$\arg(z \cdot w) = \arg z + \arg w.$$

   Here, $\arg z_1 = \arg z_2$ if the principal argument of z_1 differs from that of z_2 by a multiple of 2π.

   (b) By induction on $n \in \mathbb{N}$, prove De Moivre's Theorem:

   $$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi.$$

   (c) Use De Moivre's Theorem to derive the formulae for $\cos 3\varphi$ and $\sin 3\varphi$ in terms of $\cos\varphi$ and $\sin\varphi$.

6. (a) Compute the square roots of $z = -1 - i$,
   (b) Compute the cube roots of $z = -8$

7. Find all $z \in \mathbb{C}$ such that $z^2 \in \mathbb{R}$