LINEAR SYSTEMS AND PROPERTIES OF FUNCTIONS

The questions on this first page focus on direct computations. On the next page, try setting up a system of linear equations to solve questions 5 and 6, using the given information.

1. Are the following functions injective, surjective or bijective? In the latter case, find the inverse. Justify your answers.

   (a) $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad f(x) = |x|,$

   (b) $g : \mathbb{N} \to \mathbb{N}, \quad g(n) = n + 1,$

   (c) $h : \mathbb{N}_0 \to \mathbb{Z}, \quad h(n) = \begin{cases} \frac{n}{2}, & \text{for } n \text{ even.} \\ -\frac{n+1}{2}, & \text{for } n \text{ odd.} \end{cases}$

   Above, $\mathbb{N}$ denotes the set of natural numbers excluding zero; in $\mathbb{N}_0$ zero is included.

2. Solve the following linear systems via elimination. Sketch the solutions to (b) graphically, as an intersection of lines in the x-y-plane.

   (a) $\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 8 \\ x & + & 3y & + & 3z & = & 10 \\ x & + & 2y & + & 4z & = & 9 \end{array}\right|$

   (b) $\left|\begin{array}{ccccc} x & - & 2y & = & 2 \\ 3x & + & 5y & = & 17 \end{array}\right|$

   (c) $\left|\begin{array}{ccccccc} x & + & 4y & + & z & = & 0 \\ 4x & + & 13y & + & 7z & = & 0 \\ 7x & + & 22y & + & 13z & = & 1 \end{array}\right|$

   Now, solve this exercise using matrices.

3. Consider the linear system

$$\left|\begin{array}{ccccccc} x & + & y & - & z & = & -2 \\ 3x & - & 5y & + & 13z & = & 18 \\ x & - & 2y & + & 5z & = & k \end{array}\right|$$

   where k is an arbitrary constant.

   (a) For which value(s) of k does this system have one or infinitely many solutions?

   (b) For each of these values, how many solutions does the system have?

   (c) Write down all solutions.

4. Why are linear systems particularly easy to solve when they are in triangular form? Answer by considering the upper triangular system

$$\left|\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & x_3 & + & 4x_4 & = & -3 \\ & & x_2 & + & 3x_3 & + & 7x_4 & = & 5 \\ & & & & x_3 & + & 2x_4 & = & 2 \\ & & & & & & x_4 & = & 0 \end{array}\right|.$$

5. We call a function f a polynomial of degree 2 if it is of the form $f(t) = at^2+bt+c$, with $a \neq 0$. Find the polynomial of degree 2 whose graph passes through the points $(-1, 1)$, $(2, 3)$ and $(3, 13)$ in the x-y-plane.

6. We assume that parking meters in Zürich only accept coins of 20ct, 50ct and 1 Fr. As an incentive, the city council offers a reward to any police who, from their daily round, brings back exactly 1000 coins altogether worth 1000 CHF. Determine the possible coin combinations for which this reward can be claimed.

7. Let
$$A = \begin{pmatrix} 0 & -1 & 2 \\ 4 & 11 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 6 & 1 \end{pmatrix}.$$
Compute AB and BA.