DIFFERENTIAL CALCULUS

1. (a) $(x^8 e^{-x^3} - x - 100)' = 8x^7 e^{-x^3} - x^8 \cdot (3x^2 e^{-x^3}) - 1 = x^7 e^{-x^3}(8 - 3x^3) - 1$

   (b)

$$\left(\frac{\ln(\sin^2(x))}{\cos(x)}\right)' = \frac{\frac{1}{\sin^2 x} \cdot (2 \sin x \cos^2 x) + \ln(\sin^2 x) \sin x}{\cos^2 x}$$
$$= \frac{2}{\sin x} + \frac{\ln(\sin^2 x) \sin x}{\cos^2 x}$$

   (c) Set $y = \arctan(\sqrt{x})$ such that $\tan y = \sqrt{x}$. By implicitly differentiating with respect to x and using that $\tan(x)' = (\cos^2(x))^{-1}$, we have

$$\sec^2 y \cdot y' = \frac{1}{2\sqrt{x}}.$$

   In particular, since $\sec^2 y = 1 + \tan^2 y$ where $\sec y = \frac{1}{\cos y}$, we can write

$$y' = \arctan(\sqrt{x})' = \frac{1}{2\sqrt{x} \cdot \sec^2 y} = \frac{1}{2\sqrt{x}(1 + \tan^2 y)} = \frac{1}{2\sqrt{x}(1 + x)}.$$

2. Proceed by using the chain rule repeatedly:

   (a)

$$f'(x) = e^{\sin(x^3 + \cos x^2)} \left(\sin(x^3 + \cos x^2)\right)'$$
$$= e^{\sin(x^3 + \cos x^2)} \left(\cos(x^3 + \cos x^2)\right) (3x^2 - \sin x^2 \cdot (x^2)')$$
$$= e^{\sin(x^3 + \cos x^2)} \left(\cos(x^3 + \cos x^2)\right) (3x^2 - \sin x^2 \cdot 2x).$$

   (b)

$$g'(x) = 2 \cos\left(\frac{x^3 + 1}{x^2 + 1}\right) \cdot \left(\cos\left(\frac{x^3 + 1}{x^2 + 1}\right)\right)'$$
$$= -2 \cos\left(\frac{x^3 + 1}{x^2 + 1}\right) \sin\left(\frac{x^3 + 1}{x^2 + 1}\right) \left((x^3 + 1)(x^2 + 1)^{-1}\right)'$$
$$= -2 \cos\left(\frac{x^3 + 1}{x^2 + 1}\right) \sin\left(\frac{x^3 + 1}{x^2 + 1}\right) \cdot$$
$$\left((3x^2) \cdot (x^2 + 1)^{-1} - (x^3 + 1) \cdot (x^2 + 1)^{-2} \cdot (2x)\right).$$

3. Recall that the derivative of a function at a point x_0 is precisely the slope of the tangent line at this point. Therefore, the graph of f has a horizontal tangent at points x_0 for which $f'(x_0) = 0$. We compute

$$f'(x) = (e^{\sin x + \cos x})' = e^{\sin x + \cos x}(\cos x - \sin x).$$

It follows that $f'(x) = 0$ exactly when $\cos x = \sin x$. In the main interval $[0, 2\pi)$, this happens at the points $x_1 = \frac{\pi}{4}$ and $x_2 = \frac{5\pi}{4}$. In $\mathbb{R}$, the solutions are given by $x = \frac{\pi}{4} + k\pi$ for any integer k.

4. The first derivative of h is given by

$$h'(x) = \frac{1}{x \ln x},$$

and so we compute the second derivative to be

$$\begin{aligned} h''(x) &= -\frac{(\ln x + 1)}{(x \ln x)^2} \\ &= -\left(\frac{1}{x^2 \ln x} + \frac{1}{(x \ln x)^2}\right) \\ &= \frac{-1}{x^2 \ln x}\left(1 + \frac{1}{\ln x}\right). \end{aligned}$$

5. By the mean value theorem, $f(1) - f(0) = f(\varepsilon)$ for some $\varepsilon \in [0, 1]$. Since $f'(\varepsilon) \leq 2$ by assumption, it follows that $f(1) \leq 2 + f(0) = 1$. Equality holds for the function $f(x) = -1 + 2x$.

6. Apply the quotient rule, product rule and chain rule repeatedly:

$$\begin{aligned} \left(\frac{f(x^3)}{xf(x^2)}\right)' &= \frac{f(x^3)' \cdot xf(x^2) - f(x^3) \cdot (xf(x^2))'}{(xf(x^2))^2} \\ &= \frac{3x^2 f'(x^3) \cdot xf(x^2) - f(x^3) \cdot (f(x^2) + 2x^2 f'(x^2))}{(xf(x^2))^2} \\ &= \frac{3x^3 f'(x^3) f(x^2) - f(x^3) f(x^2) - 2x^2 f(x^3) f'(x^2))}{(xf(x^2))^2}. \end{aligned}$$