LINEAR (IN)DEPENDENCE AND MATRICES

1. We use the augmented matrices

$$\left(\begin{array}{ccc|ccc} 1 & 0 & t & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & t & 1 & 0 & 0 \\ 0 & 1 & -2t & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & t & 1 & 0 & 0 \\ 0 & 1 & -2t & -2 & 1 & 0 \\ 0 & 0 & 1+2t & 2 & -1 & 1 \end{array}\right)$$

If $t = -1/2$, then the matrix on the left is triangular, but one of its coefficients on the diagonal is zero. Thus, the matrix is not invertible. Otherwise, the matrix is invertible. We continue applying the Gauss pivot method to determine its inverse.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & t & 1 & 0 & 0 \\ 0 & 1 & -2t & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{1+2t} & \frac{-1}{1+2t} & \frac{1}{1+2t} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{1+2t} & \frac{t}{1+2t} & \frac{-t}{1+2t} \\ 0 & 1 & 0 & \frac{-2}{1+2t} & \frac{1}{1+2t} & \frac{2t}{1+2t} \\ 0 & 0 & 1 & \frac{2}{1+2t} & \frac{-1}{1+2t} & \frac{1}{1+2t} \end{array}\right)$$

Thus the inverse of the matrix is

$$\begin{pmatrix} \frac{1}{1+2t} & \frac{t}{1+2t} & \frac{-t}{1+2t} \\ \frac{-2}{1+2t} & \frac{1}{1+2t} & \frac{2t}{1+2t} \\ \frac{2}{1+2t} & \frac{-1}{1+2t} & \frac{1}{1+2t} \end{pmatrix}$$

2. (a) The vectors $\vec{v}_1$, $\vec{v}_2$ are linearly dependent if and only if the equation

$$a\vec{v}_1 + b\vec{v}_2 = \vec{0}$$

has a non-zero solution for real numbers a and b. Introducing the values of $\vec{v}_1$ and $\vec{v}_2$, this reads

$$a\begin{pmatrix} 3 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The second row implies that $b = 0$; after substituting this into $3a + b = 0$, we conclude that $a = 0$. Therefore, $(0, 0)$ is the unique solution, and the vectors $\vec{v}_1$ and v_2 are linearly *independent*.

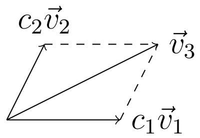
For the vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$, we want to find a non-zero solution to

$$a\begin{pmatrix}3\\0\end{pmatrix} + b\begin{pmatrix}1\\2\end{pmatrix} + c\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

One solution is $\vec{v}_1 + v_2 - 2v_3 = \vec{0}$, and so the vectors are linearly *dependent*. We can rewrite the solution as $\vec{v}_3 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$.

(b) We may rewrite $a\vec{v}_1 + b\vec{v}_2 = \vec{0}$ as $\vec{v}_1 = -\frac{b}{a}\vec{v}_2$, so the vectors $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent if and only if one is a scalar multiple of the other. This happens precisely when the two vectors lie on the same line. In the picture this is clearly not the case, thus the vectors must be linearly independent.

However, $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly dependent, as with a correct scaling of $\vec{v}_1$ and $\vec{v}_2$, we get



As calculated in (a), the correct scaling is given by $c_1 = c_2 = \frac{1}{2}$.

3. (a)
- $(1,1,1)$ and $(0,1,-2)$ are linearly independent because $\lambda \cdot 0 = 0$ for all $\lambda \in \mathbb{R}$, which implies that

$$\lambda(0,1,-2) = (0,\lambda,-2\lambda) \neq (1,1,1) \text{ for all } \lambda \in \mathbb{R}.$$

- The vectors are linearly dependent, because $(1,1,1) = (1,1,0) - (0,0,-1)$.

- $(1,1,1)$, $(1,1,0)$ and $(1,0,-1)$ are linearly independent, since

$$(1,1,1) \neq \lambda(1,1,0) + \mu(1,0,-1) = (\lambda+\mu, \lambda, -\mu) \text{ for all } \lambda, \mu \in \mathbb{R}.$$

(b) The question amounts to finding all non-trivial solutions $(\lambda_1, \lambda_2, \lambda_3)$ of the equation

$$\lambda_1(1,1,1) + \lambda_2(1,1,0) + \lambda_3(t,0,-1) = (0,0,0),$$

for $t \in \mathbb{R}$. This is equivalent to solving the linear system

$$\left|\begin{array}{ccccccc} \lambda_1 & + & \lambda_2 & + & \lambda_3 t & = & 0 \\ \lambda_1 & + & \lambda_2 & & & = & 0 \\ \lambda_1 & & & - & \lambda_3 & = & 0 \end{array}\right|.$$

In particular, we see that $\lambda_1 = -\lambda_2$ and $\lambda_1 = \lambda_3$. Substituting this in back in the first row, we have

$$\lambda_1 + \lambda_2 + \lambda_3 t = \lambda_1 - \lambda_1 + \lambda_1 t = \lambda_1 t = 0;$$

the condition of linear dependence now implies that the only possible solution is $t = 0$. In other words, the vectors are linearly dependent if and only if $t = 0$.

4. (a) All triangular matrices with zeros on the diagonal are nilpotent, for example

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

You can easily check that $A^3 = 0$.

(b) Suppose that

$$\vec{0} = c_0\vec{x} + c_1 A\vec{x} + c_2 A^2\vec{x} + \cdots c_{m-1}A^{m-1}\vec{x}$$

for some $c_0, c_1, \cdots, c_{m-1} \in \mathbb{R}$. Fix $0 \leq i \leq m-1$ and suppose further that all $c's$ before c_i are zero. The above equation then reads

$$\vec{0} = c_i A^i\vec{x} + c_{i+1}A^{i+1}\vec{x} + c_{i+2}A^{i+2}\vec{x} + \cdots c_{m-1}A^{m-1}\vec{x}$$

(if $i = 0$, then equation remains unchanged). Multiply this by A^{m-i-1} to find

$$\begin{aligned} \vec{0} &= c_i A^{m-1}\vec{x} + c_{i+1}A^m\vec{x} + \cdots c_{m-1}A^{2m-i-2}\vec{x} \\ &= c_i A^{m-1}\vec{x} + A^m(c_{i+1}\vec{x} + c_{i+2}A\vec{x} + \cdots c_{m-1}A^{m-i-2}\vec{x}) \\ &= c_i A^{m-1}\vec{x}, \end{aligned}$$

since $A^m = 0$. By choice $A^{m-1}\vec{x} \neq 0$, and thus $c_i = 0$. Since this argument holds for any $0 \leq i \leq m-1$, we conclude that all $c_0, \cdots, c_{m-1}$ are zero and the vectors $A^1\vec{x}, A^2\vec{x}, \cdots, A^{m-1}\vec{x}$ are linearly independent.

5.

(a) $\begin{pmatrix} 4 & 6 \\ 3 & 4 \end{pmatrix}$, (b) not possible,

(c) $\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{pmatrix}$,

(e) $\begin{pmatrix} b \\ e \\ h \end{pmatrix}$, (f) $\begin{pmatrix} a & b \\ c & d \\ 0 & 0 \end{pmatrix}$.

6. (a) The corresponding system of equations is given by

$$\left| \begin{array}{ccccc} x_1 & + & 2x_2 & = & 2 \\ 3x_1 & + & 4x_2 & = & 1 \end{array} \right|,$$

and we compute the unique solution to be $\vec{x} = \left(\begin{smallmatrix} -3 \\ 5/2 \end{smallmatrix}\right)$.

(b) Assume first that $a \neq 0$, and use the standard algorithm for inverting matrices as presented in the lectures. Below is a suggestion of steps:

i. multiply row 1 by $\frac{1}{a}$,
ii. add $(-c) \cdot$ (row 1) to row 2,
iii. multiply row 2 by $\frac{a}{ad-bc}$,
iv. add $\left(-\frac{b}{a}\right) \cdot$ (row 2) to row 1.

The result is

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix};$$

multiplying this matrix with A gives $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now assume that $a = 0$. Since $ad - bc \neq 0$, we know that $bc \neq 0$ and it holds that

$$-\frac{1}{bc}\begin{pmatrix} d & -b \\ -c & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = I$$

In particular, the matrix A is invertible.

(c) Since $4 - 6 \neq 0$, the matrix A is invertible and

$$\vec{x} = A^{-1}\vec{b} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \\ 5/2 \end{pmatrix}.$$

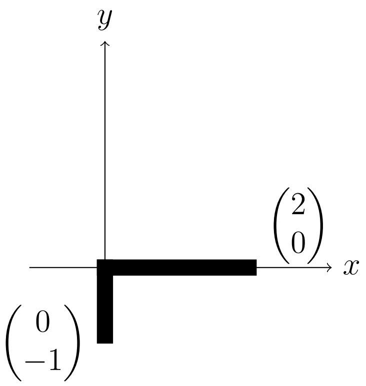
7. The matrices A, B and C have the following effects on any vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}, \quad B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

The matrix A scales vectors by a multiple of 3. Thus after the action of A, the letter L still sits at the origin, but has become three times larger: it extends to $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ on the vertical axis, and to $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ on the horizontal one.The inverse transformation is simply a rescaling by 1/3. Using the formula from 5.(c), you can check that the inverse matrix is given by $\begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$.

The matrix B projects points onto the horizontal axis. Under this projection, the letter L is reduced to the segment $[0, 1]$ on the x-axis. Intuitively, there should be no well-defined inverse, since any point on the vertical line through $(x, 0)$ is a potential pre-image of the transformation. Applying 5.(b), you can check that the criterion for inverses is not fulfilled by the matrix B.

To understand the geometric action of the matrix C, it may be easier to look at its effect on the letter L :



The letter L has been rotated by 90^o clockwise. The inverse is the rotation of 90^o counterclockwise, given by $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

8. (a) The determinant of an upper triangular (or lower triangular) $n \times n$-matrix is simply the product of the main diagonal entries: to see this, expand along the column with $n-1$ zeros. In particular, $\det A = 1 \times 5 \times 8 \times 3 = 120$.

   (b) Notice that the second row is just a multiple of the first row. Thus, since the columns are *not* linearly independent, $\det B = 0$.

9. We expand along the row staring with x to deduce that

$$\det\left[\begin{pmatrix}1 & 1 & 2 & 3 & 4\\ 9 & 0 & 2 & 3 & 4\\ 9 & 0 & 0 & 3 & 4\\ x & 1 & 2 & 9 & 1\\ 7 & 0 & 0 & 0 & 4\end{pmatrix}\right] = -x \cdot \det\left[\begin{pmatrix}1 & 2 & 3 & 4\\ 0 & 2 & 3 & 4\\ 0 & 0 & 3 & 4\\ 0 & 0 & 0 & 4\end{pmatrix}\right] + C,$$

for some $C \in \mathbb{R}$. Since the determinant is just a real number, we deduce that

$$f'(x) = -\det\left[\begin{pmatrix}1 & 2 & 3 & 4\\ 0 & 2 & 3 & 4\\ 0 & 0 & 3 & 4\\ 0 & 0 & 0 & 4\end{pmatrix}\right] = -24;$$

we quickly computed the determinant by using the same reasoning as in q.1(a).

10. Denote the two vectors constituting the basis $\mathcal{B}$ by $b_1 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), b_2 = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$. Then

$$f_1(b_1) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = b_1 = 1 \cdot b_1 + 0 \cdot b_2,$$

$$f_1(b_2) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = -b_2 = 0 \cdot b_1 - 1 \cdot b_2,$$

and so the required matrix B is given by

$$B = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$