LINEAR ALGEBRA AND SOME APPLICATIONS

1. The characteristic polynomial $p(\lambda)$ of A_k is given by

$$\begin{aligned} p(\lambda) = \det(A_k - \lambda I_2) &= \det\left[\begin{pmatrix} -1-\lambda & k \\ 4 & 3-\lambda \end{pmatrix}\right] \\ &= (-1-\lambda)(3-\lambda) - 4k \\ &= \lambda^2 - 2\lambda - 3 - 4k. \end{aligned}$$

If $\lambda = 5$ is an eigenvalue, then $p(5) = 0$; in other words,

$$5^2 - 2 \cdot 5 - 3 - 4k \stackrel{!}{=} 0.$$

Thus, after solving for k, we deduce that 5 is an eigenvalue of A_k if and only if $k = 3$.

2. (a) Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be any 2×2-matrix. Then the corresponding characteristic polynomial $p(\lambda)$ is given by

$$\begin{aligned} p(\lambda) = \det(A - \lambda I_2) &= \det\left[\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\right] \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda - (ad - bc) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A). \end{aligned}$$

   (b) Since both the trace and the determinant of a matrix are invariant under a change of basis, we have $\det(A) = \lambda_1 \cdot \lambda_2$ and $\operatorname{tr}(A) = \lambda_1 + \lambda_2$.

3. (a) The characteristic polynomial of the matrix A is

$$\begin{aligned} \det(A - \lambda I_3) &= \det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \\ &= -\lambda(1-\lambda)^2 - 2(1-\lambda) \\ &= (1-\lambda)(\lambda^2 - \lambda - 2), \end{aligned}$$

   and so the eigenvalues are given by

$$\lambda_1 = 1\,, \quad \lambda_2 = -1\,, \quad \lambda_3 = 2\,.$$

   An eigenvector $\vec{v_1}^\intercal = (x_1, x_2, x_3)^\intercal$ corresponding to the eigenvalue λ_1 must satisfy

$$(A - \lambda_1 I_3)\vec{v_1} = 0.$$

In other words, after solving the system of linear equations

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

we find that $\vec{v_1}^\intercal = (1,-1,0)^\intercal$. Similarly, we compute $\vec{v_2}^\intercal = (1,1,-2)^\intercal$ and $\vec{v_3}^\intercal = (1,1,1)^\intercal$.

(b) Consider the matrix S whose columns are the eigenvectors of A,

$$S = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}.$$

Then

$$S^{-1}AS = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. (a) The map f_3 describes an anticlockwise rotation of the plane $\mathbb{R}^2$ by an angle of $\vartheta \in [0, 2\pi)$. To understand this better, try computing the matrix product $A\vec{v}$ for some specific vectors $\vec{v} \in \mathbb{R}^2$. For example,

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\vartheta - \sin\vartheta \\ \sin\vartheta + \cos\vartheta \end{pmatrix};$$

then $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 \\ \sqrt{2} \end{smallmatrix}\right)$ when $\vartheta = \frac{\pi}{4}$ and $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right)$ when $\vartheta = \pi$.

(b) We calculate the characteristic polynomial of A to be

$$\det\left[\begin{pmatrix} \cos\vartheta - \lambda & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta - \lambda \end{pmatrix}\right] = (\cos\vartheta - \lambda)^2 + \sin^2\vartheta.$$

The equation $(\cos\vartheta - \lambda)^2 + \sin^2\vartheta = 0$ has no real solutions, except when ϑ is a multiple of π such that A represents the identity matrix or a reflection about the origin. To find the remaining eigenvalues, we view A as a matrix over the complex numbers $\mathbb{C}$, such that f_3 describes a linear transformation of $\mathbb{C}^2$. Thus,

$$\lambda_1 = \cos\vartheta + i\sin\vartheta = e^{i\vartheta}$$
$$\lambda_2 = \cos\vartheta - i\sin\vartheta = e^{-i\vartheta}.$$

To find an eigenvector $\vec{v_1}^\intercal = (x_1, x_2)^\intercal$ corresponding to λ_1, we solve the matrix equation $(A - I\lambda_1)\vec{v} = 0$, or more explicitly

$$\begin{pmatrix} -i\sin\vartheta & -\sin\vartheta \\ \sin\vartheta & -i\sin\vartheta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The generic solution of this system is $(x_1, x_2)^\intercal = (z, -iz)^\intercal$ for any $z \in \mathbb{C}$. Similarly, we compute the eigenvectors of λ_2 to be of the form $(x_1, x_2)^\intercal = (z, iz)^\intercal$ for $z \in \mathbb{C}$.

(c) Using the result of q.1(b) with $z = 1$, the effect of f_3 on the two linearly independent eigenvectors $b_1 = \left(\begin{smallmatrix}1\\-i\end{smallmatrix}\right), b_2 = \left(\begin{smallmatrix}1\\i\end{smallmatrix}\right)$ is

$$f_3(b_1) = \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix} = \begin{pmatrix}\cos\vartheta + i\sin\vartheta\\ -i\cos\vartheta + \sin\vartheta\end{pmatrix} = e^{i\vartheta}\cdot b_1.$$

$$f_3(b_2) = \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}\cos\vartheta - i\sin\vartheta\\ i\cos\vartheta + \sin\vartheta\end{pmatrix} = e^{-i\vartheta}\cdot b_2,$$

The required matrix B is therefore given by

$$B = \begin{pmatrix}e^{i\vartheta} & 0\\ 0 & e^{-i\vartheta}\end{pmatrix}.$$

5. (a) The initial system

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 1\\ 2x & + & 3y & + & \alpha z & = & 3\\ x & + & \alpha y & + & 3z & = & 2\end{array}$$

is equivalent to

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 1\\ & & y & + & (\alpha+2)z & = & 1\\ & & & + & (\alpha+3)(2-\alpha)z & = & 2-\alpha.\end{array}$$

(b) If $(\alpha+3)(2-\alpha)\neq 0$, the unique solution (x, y, z) is given by

$$\begin{aligned} x &= 1 - \left(1 - \frac{\alpha+2}{\alpha+3}\right) + \frac{1}{\alpha+3} = 1,\\ y &= 1 - \frac{\alpha+2}{\alpha+3} = \frac{1}{\alpha+3},\\ z &= \frac{1}{\alpha+3}.\end{aligned}$$

If $\alpha = -3$, the third row in the above system is 0=5, so no solutions exist. If $\alpha = 2$, there are infinitely many solutions parametrised by

$$(x, y, z) = (0, 1, 0) + z(5, -4, 1) = (5z, 1-4z, z) \text{ for } z\in\mathbb{R} \text{ arbitrary.}$$

(c) Using the results in q.1(a) of problem set 10, we easily compute the determinant to be $(\alpha+3)(2-\alpha)$. It vanishes for $\alpha = -3$ and $\alpha = 2$. Whenever A has vanishing determinant, one of its eigenvalues must be zero. In particular, the kernel cannot be trivial, and so A projects the whole space into a lower dimension.

6. (a) The rank of B is simply the dimension of the entire space minus the dimension of the kernel. Since 0 is an eigenvalue, $\ker B$ is at least 1-dimensional. However, the other two eigenvalues are non-zero, thus we conclude that B has rank 2.

(b) The matrix B is non-invertible, thus has determinant zero:

$$\det(B^\intercal B) = \det(B^\intercal) \cdot \det(B) = 0.$$

(c) The eigenvalues of $B^\intercal B$ cannot be determined using the given information. Consider the following justification. The matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

both have the same eigenvalues, yet

$$A^\intercal A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad B^\intercal B = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$$

do not have the same eigenvalues.

(d) Since eigenvectors corresponding to distinct eigenvalues are linearly independent, there exists a basis (an eigenbasis) in which B has the representation

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $B^2 + I$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

and so the eigenvalues of the inverse are easily determined to be $1, \frac{1}{2}, \frac{1}{5}$.

(e) Since the trace is invariant under a change of basis, we use the results of (d) to compute the trace to be $1 + \frac{1}{2} + \frac{1}{5} = \frac{17}{10}$ (the sum of the eigenvalues of $(B^2 + I)^{-1}$ in an eigenbasis).

7. (a) **Procedure presented in the lectures**

The system

$$\begin{aligned} \dot{x}_1 &= 5x_1 + 4x_2 \\ \dot{x}_2 &= 3x_1 - 6x_2 \end{aligned}$$

can be rewritten as $\dot{\vec{x}} = A\vec{x}$, where

$$A = \begin{pmatrix} 5 & 4 \\ 3 & -6 \end{pmatrix}, \quad \text{and} \quad \vec{x} = \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

We first transform A into a triangular or a diagonal matrix. The characteristic polynomial is

$$p_A(\lambda) = (5 - \lambda)(-6 - \lambda) - 12,$$

and by solving this quadratic equation we determine the two eigenvalues to be $\lambda_1 = -7$, $\lambda_2 = 6$. Two corresponding eigenvectors are then $\vec{v}_1 = (1, -3)^\intercal$ and $\vec{v}_2 = (4, 1)^\intercal$, respectively. Thus, in an eigenbasis, A has the representation

$$D = T^{-1}AT = \begin{pmatrix} -7 & 0 \\ 0 & 6 \end{pmatrix}, \text{ where } \quad T = \begin{pmatrix} 1 & 4 \\ -3 & 1 \end{pmatrix}.$$

Now, the solution of the system

$$\begin{aligned} \dot{y}_1 &= -7y_1 \\ \dot{y}_2 &= 6y_2 \end{aligned}$$

is given by $y_1(t) = C_1 e^{-7t}$ and $y_2(t) = C_2 e^{6t}$ for some constants $C_1, C_2 \in \mathbb{R}$. In particular

$$\begin{aligned} x_1 &= y_1 + 4y_2 = C_1 e^{-7t} + 4C_2 e^{6t} \\ x_2 &= -3y_1 + y_2 = -3C_1 e^{-7t} + C_2 e^{6t}. \end{aligned}$$

Subject to the initial conditions $x(0) = 13$ and $y(0) = 0$, we have the equations

$$\begin{aligned} 13 &= C_1 + 4C_2 \\ 0 &= -3C_1 + C_2, \end{aligned}$$

from which we easily determine that $C_1 = 1$ and $C_2 = 3$.
The final solution is therefore

$$\begin{aligned} x_1(t) &= e^{-7t} + 12e^{6t} \\ x_2(t) &= -3e^{-7t} + 3e^{6t}. \end{aligned}$$

(b) **Without a direct change of basis**

In case the matrix A has two real and distinct eigenvalues λ_1 and λ_2 as above, the computations tell us that the general solution $\vec{x} = (x_1, x_2)^\intercal$ to any such system is given by

$$\vec{x} = C_1 \vec{v}_1 e^{\lambda_1 t} + C_2 \vec{v}_2 e^{\lambda_2 t},$$

where $\vec{v}_1, \vec{v}_2$ are two corresponding eigenvectors and the constants C_1, C_2 are determined by the initial conditions. In other words, to solve such a system it suffices to find the eigenvalues and corresponding eigenvectors, then apply the initial conditions.

8. We write the system

$$\begin{aligned} \dot{x}_1 &= -3x_1 + 2x_2 \\ \dot{x}_2 &= -8x_1 + 5x_2 \end{aligned}$$

in matrix form $\dot{\vec{x}} = A\vec{x}$, where

$$A = \begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix} \quad \text{and} \quad \vec{x} = \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

The characteristic polynomial of A is

$$p_A(\lambda) = (-3-\lambda)(5-\lambda) + 16,$$

and so A has repeated eigenvalue $\lambda = \lambda_{1,2} = 1$. Since we require two linearly independent eigenvectors to form a general solution, we need to modify our standard procedure. (Recall that we run into similar issues with repeated roots when solving second order ODE's.)

We make the ansatz $\vec{x} = \vec{v}_1 e^t + \vec{v}_2 te^t$ for some vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ independent of the parameter t. Using the product rule, we determine

$$\dot{\vec{x}} = (\vec{v}_1 + \vec{v}_2)e^t + \vec{v}_2 te^t = A\vec{x} = A\vec{v}_1 e^t + A\vec{v}_2 te^t.$$

Notice that this equation can be separated into one corresponding to e^t, and one corresponding to te^t:

$$\begin{aligned} \vec{v}_1 + \vec{v}_2 &= A\vec{v}_1 \\ \vec{v}_2 &= A\vec{v}_2. \end{aligned}$$

From the second equation we easily compute

$$(A - I_2)\vec{v}_2 = \left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \vec{v}_2 = \begin{pmatrix} -4 & 2 \\ -8 & 4 \end{pmatrix} \vec{v}_2 = 0,$$

and the solutions are given by $\vec{v}_2 = (C_2, 2C_2)^\intercal$ for $C_2 \in \mathbb{R}$. Using this result in the first equation, we set up the system

$$(A - I)\vec{v}_1 = \left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \vec{v}_1 = \begin{pmatrix} -4 & 2 \\ -8 & 4 \end{pmatrix} \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{v}_2 \quad \text{for } C_2 = 1.$$

The solutions are $\vec{v}_1 = (C_1, \frac{1}{2} + 2C_1)^\intercal$ for $C_1 \in \mathbb{R}$.

Altogether, the general solution to our system of ODE's is therefore given by

$$\begin{aligned} x_1(t) &= C_1 e^t + C_2 te^t \\ x_2(t) &= (\tfrac{1}{2} + 2C_1)e^t + 2C_2 te^t. \end{aligned}$$

We determine the constants C_1, C_2 using the initial conditions $x_1(0) = 0$ and $x_2(0) = 1$: since the system

$$\begin{aligned} 0 &= C_1 \\ 1 &= \tfrac{1}{2} + 2C_1 \end{aligned}$$

has no solution, we conclude that no solutions $x_1(t), x_2(t)$ subject to these conditions exist.

You may notice, if trying to solve another system with repeated eigenvalue, that the above ansatz does not work in general. Instead, one should try the guess $\vec{x} = C_1\vec{v}e^{\lambda t} + C_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w})$, where $\vec{v}$ is an eigenvector corresponding to the repeated root λ, and $\vec{w}$ satisfies the equation $(A - \lambda I)\vec{w} = \vec{v}$. The constants $C_1, C_2 \in \mathbb{R}$ are determined by the initial conditions. We discuss this in exercise class 11.

9. The gradient of f is $\nabla f(x, y, z) = (6xy^2, 6x^2y, 10z)$, whilst he Hessian matrix is given by
$$\operatorname{Hess}(f) = \begin{pmatrix} 6y^2 & 12xy & 0 \\ 12xy & 6x^2 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

10. The gradient of T is given by $\nabla T = (2x - 1, 4y)$, and so $\nabla T = \vec{0}$ at $(\frac{1}{2}, 0)$. In particular,
$$\operatorname{Hess} T(\tfrac{1}{2}, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$
which is positive-definite; the point $(\frac{1}{2}, 0)$ is therefore a local minimum at temperature $T(\frac{1}{2}, 0) = -\frac{1}{4}$. It remains to check the behaviour of T at the boundary of D, described by the equation $x^2 + y^2 = 1$. There, T has the representation $T(x, y)|_{\partial D} = g(x) = x^2 + 2(1 - x^2) - x$, with derivative $g'(x) = 2x - 4x - 1 = -2x - 1$. We compute that a maximum of g occurs at $x = -\frac{1}{2}$, and $g(-\frac{1}{2}) = \frac{9}{4}$. Finally, since $T(-1, 0) = 2$ and $T(1, 0) = 0$, we deduce the following:

    The coldest point $(\frac{1}{2}, 0)$ of D has temperature $-\frac{1}{4}$, whilst the hottest point $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ has temperature $\frac{9}{4}$.

11. The function is zero on the coordinate axes and positive elsewhere.