MATHEMATICS

Problem Set 2

DIFFERENTIAL CALCULUS

1. (a) We have to solve the equation $f(x) = g(x)$ and compute that

$$\begin{aligned} & f(x) = g(x) \\ \iff & 4x^3 + 2x^2 - 5x - 2 = 2x^2 - x - 2 \\ \iff & 4x^3 - 4x = 0 \\ \iff & 4x(x^2 - 1) = 0 \\ \iff & 4x(x+1)(x-1) = 0 \\ \iff & (x+1)x(x-1) = 0. \end{aligned}$$

Therefore, we get as result

$$x_1 = -1,\ x_2 = 0,\ x_3 = 1.$$

(b) The local minimum and the local maximum must satisfy the conditions

$$\begin{aligned} f'(x_{\min}) = 0 \text{ and } f''(x_{\min}) > 0, \\ f'(x_{\max}) = 0 \text{ and } f''(x_{\max}) < 0. \end{aligned}$$

Therefore, we must solve the equation $f'(x) = 0$ and calculate

$$\begin{aligned} f'(x) &= (4x^3 + 2x^2 - 5x - 2)' \\ &= 4(x^3)' + 2(x^2)' - 5x' - 2' \\ &= 4 \cdot 3x^2 + 2 \cdot 2x - 5 - 0 \\ &= 12x^2 + 4x - 5 \\ &= (2x - 1) \cdot (6x + 5) \\ &= 12(x - \tfrac{1}{2}) \cdot (x + \tfrac{5}{6}) = 0. \end{aligned}$$

Thus, we obtain that $x_{\min}, x_{\max} \in \left\{\frac{1}{2}, -\frac{5}{6}\right\}$.

To decide, which point is the minimum and which point is the maximum, we look at the conditions for the second derivative $f''(x)$ and compute

$$\begin{aligned} f''(x) &= (f'(x))' \\ &= (12x^2 + 4x - 5)' \\ &= 12(x^2)' + 4x' - 5' \\ &= 12 \cdot 2x + 4 \\ &= 24x + 4 \\ &= 4(6x + 1), \end{aligned}$$

to see that

$$f''(\tfrac{1}{2}) = 4(6\tfrac{1}{2} + 1) = 4 \cdot 4 = 16 > 0,$$
$$f''(-\tfrac{5}{6}) = 4(6(-\tfrac{5}{6}) + 1) = 4 \cdot (-4) = -16 < 0.$$

Therefore, we conclude that $x_{\min} = \frac{1}{2}$ and $x_{\max} = -\frac{5}{6}$.

2. We first find the critical points (the local extrema), that is, the points x for which $f'(x) = 0$. The derivative is

$$\begin{aligned} f'(x) &= (x^4 - 4x^3 + 4x^2 - 3)' \\ &= (x^4)' - 4(x^3)' + 4(x^2)' - 3' \\ &= 4x^3 - 4 \cdot 3x^2 + 4 \cdot 2x - 0 \\ &= 4x^3 - 12x^2 + 8x \\ &= 4x(x^2 - 3x + 2) \\ &= 4x(x-1)(x-2). \end{aligned}$$

Hence the local extrema occur at $x_1 = 0$, $x_2 = 1$ and $x_3 = 2$.
To determine whether the f-values at these three points are local minima or local maxima, we look at the second derivative, which is given by

$$\begin{aligned} f''(x) &= (f'(x))' \\ &= (4x^3 - 12x^2 + 8x)' \\ &= 4(x^3)' - 12(x^2)' + 8x' \\ &= 4 \cdot 3x^2 - 12 \cdot 2x + 8 \\ &= 12x^2 - 24x + 8 \\ &= 4(3x^2 - 6x + 2). \end{aligned}$$

Because we have that

$$\begin{aligned} f''(0) &= 4(3 \cdot 0^2 - 6 \cdot 0 + 2) = 8 > 0, \\ f''(1) &= 4(3 \cdot 1^2 - 6 \cdot 1 + 2) = -4 < 0, \\ f''(2) &= 4(3 \cdot 2^2 - 6 \cdot 2 + 2) = 8 > 0, \end{aligned}$$

we have that $x = 0$ and $x = 2$ correspond to local minima and that $x = 1$ corresponds to a local maximum.

To find global extrema on the interval $[-2, 3]$, we compare the values of f at the points $0, 1, 2$ with the two endpoints $e_1 = -2$ and $e_2 = 3$ of the interval $[-2, 3]$, over which the global extrema should be identified.
Putting everything together, we have that

$$\begin{aligned} f(-2) &= (-2)^4 - 4 \cdot (-2)^3 + 4 \cdot (-2)^2 - 3 = 16 + 32 + 16 - 3 = 61, \\ f(0) &= 0^4 - 4 \cdot 0^3 + 4 \cdot 0^2 - 3 = -3, \\ f(1) &= 1^4 - 4 \cdot 1^3 + 4 \cdot 1^2 - 3 = 1 - 4 + 4 - 3 = -2, \\ f(2) &= 2^4 - 4 \cdot 2^3 + 4 \cdot 2^2 - 3 = 16 - 32 + 16 - 3 = -3, \\ f(3) &= 3^4 - 4 \cdot 3^3 + 4 \cdot 3^2 - 3 = 81 - 108 + 36 - 3 = 6. \end{aligned}$$

Hence the global maximum over the interval $[-2, 3]$ of the function $f(x) = x^4 - 4x^3 + 4x^2 - 3$ is 61 and the global minimum is -3.
These global extrema occur at the points $x_{\max} = -2$, $x_{\min}^{(1)} = 0$ and $x_{\min}^{(2)} = 2$, because we have that $f(-2) = 61$ and that $f(0) = f(2) = -3$.

3. Using the product rule and the chain rule, we compute that

$$\begin{aligned} y' &= y'(x) \\ &= \left[(x^4-1)^3 \ln(x+1)\right]' \\ &= ((x^4-1)^3)' \ln(x+1) + (x^4-1)^3 (\ln(x+1))' \\ &= 3(x^4-1)^2 \cdot (x^4-1)' \ln(x+1) + (x^4-1)^3 \frac{1}{x+1}(x+1)' \\ &= 3(x^4-1)^2(4x^3-0)\ln(x+1) + (x^4-1)^3\frac{1}{x+1} \cdot 1 \\ &= 12x^3(x^4-1)^2\ln(x+1) + \frac{(x^4-1)^3}{x+1}. \end{aligned}$$

So we have that

$$y'(0) = 12 \cdot 0^3(0^4-1)^2 \ln(0+1) + \frac{(0^4-1)^3}{0+1} = 0 + \frac{(-1)^3}{1} = (-1)^3 = -1.$$

Therefore, the tangent t to the curve y is given by $t(x) = -x$, because $t(0) = 0 = y(0)$ and $t'(0) = -1 = y'(0)$.
The normal $n(x) = mx + b$ to the curve at the origin $(x, y) = (x, y(x)) = (0, 0)$ must have a coefficient of direction $m = -\frac{1}{-1} = 1$ to make an angle of $90°$ between the lines $t(x) = -x$ and $n(x) = x + b$, so the normal is $n(x) = x + b$ for some $b \in \mathbb{R}$. Because this normal passes through the origin $(x, y) = (x, y(x)) = (0, y(0)) = (0, n(0)) = (0, 0)$, the equation for the normal must be $n(x) = x$.

4. We compute that

$$\begin{aligned}
f'(x) &= \left(\frac{e^x \sin(x)}{\cos(x)} + 1\right)' \\
&= \left(\frac{e^x \sin(x)}{\cos(x)}\right)' + 1' \\
&= \frac{(e^x \sin(x))' \cos(x) - e^x \sin(x)(\cos(x))'}{\cos^2(x)} + 0 \\
&= \frac{((e^x)' \sin(x) + e^x (\sin(x))') \cos(x) - e^x \sin(x)(-\sin(x))}{\cos^2(x)} \\
&= \frac{e^x \sin(x)\cos(x) + e^x \cos(x)\cos(x) + e^x \sin^2(x)}{\cos^2(x)} \\
&= \frac{e^x \sin(x)\cos(x) + \overbrace{e^x \cos^2(x) + e^x \sin^2(x)}^{=e^x(\sin^2(x)+\cos^2(x))=e^x}}{\cos^2(x)} \\
&= \frac{e^x \sin(x)\cos(x) + e^x}{\cos^2(x)}.
\end{aligned}$$

We see that

$$f(0) = \frac{e^0 \sin(0)}{\cos(0)} + 1 = \frac{1 \cdot 0}{1} + 1 = 1$$

and that

$$f'(0) = \frac{e^0 \sin(0)\cos(0) + e^0}{\cos^2(0)} = \frac{1 \cdot 0 \cdot 1 + 1}{1^2} = 1.$$

Therefore, we obtain that

$$\begin{aligned}
y &= f'(0)x + f(0) \\
&= x + 1.
\end{aligned}$$

5. We were told in the problem statement that the function satisfies the conditions of the Mean Value Theorem.

$$f(0) - f(-7) = f'(c)(0 - (-7))$$

So with the known values, we obtain

$$f(0) + 3 = 7f'(c).$$

Finally, let's take care of what we know about the derivative. We are told that the maximum value of the derivative is 2. So, plugging the maximum possible value of the derivative into $f'(c)$ above will, in this case, give us the maximum value of $f(0)$. Doing this gives

$$f(0) = 7f'(c) - 3 \leq 7(2) - 3 = 11.$$

So, the largest possible value for $f(0)$ is 11. Or, written as an inequality $f(0) \leq 11$.