DIFFERENTIAL CALCULUS

1. (a) Set $u := x^2 + 1$ such that $\frac{du}{dx} = 2x$.

$$\int 2x \sin(x^2+1)\,\mathrm{d}x = \int \sin u\,\mathrm{d}u = -\cos u + C = -\cos(x^2+1) + C.$$

(b) Set $u := -10x$ such that $\frac{du}{dx} = -10$.

$$\int e^{-10x}\,\mathrm{d}x = -\frac{1}{10}\int e^u\,\mathrm{d}u = -\frac{e^u}{10} + C = -\frac{e^{-10x}}{10} + C.$$

(c) Set $u := 5x - \frac{\pi}{2}$ such that $\frac{du}{dx} = 5$.

$$\int \cos(5x - \tfrac{\pi}{2})\,\mathrm{d}x = \frac{1}{5}\int \cos u\,\mathrm{d}u = \frac{\sin u}{5} + C = \frac{\sin(5x - \frac{\pi}{2})}{5} + C.$$

The definite integral from $x = 0$ to $x = \frac{\pi}{5}$ therefore evaluates to

$$\frac{\sin(5x - \frac{\pi}{2})}{5}\Bigg|_{x=0}^{x=\pi/5} = \frac{\sin\frac{\pi}{2}}{5} - \frac{\sin\frac{-\pi}{2}}{5} = \frac{2}{5}.$$

(d) Set $u := -\sqrt{x+1}$ such that $\frac{du}{dx} = -\frac{1}{2\sqrt{x+1}}$.

$$\int \frac{e^{-\sqrt{x+1}}}{\sqrt{x+1}}\,\mathrm{d}x = -2\int e^u\,\mathrm{d}u = -2e^{-\sqrt{x+1}} + C.$$

(e) Set $u := x + 3$ such that $\frac{du}{dx} = 1$.

$$\int \frac{1}{x+3}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C = \ln|x+3| + C.$$

The definite integral from $x = 2$ to $x = 7$ now evaluates to

$$\ln|x+3|\Bigg|_{x=2}^{x=7} = \ln 10 - \ln 5 = \ln 2.$$

(f) Set $u := 2x - 2$ such that $\frac{du}{dx} = 2$.

$$\int \frac{1}{2x-2}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\ln|u| + C = \ln|2x-2| + C.$$

The definite integral from $x = 3$ to $x = 5$ evaluates to

$$\frac{1}{2}\ln|2x-2|\Bigg|_{x=3}^{x=5} = \frac{1}{2}(\ln 8 - \ln 4) = \ln\sqrt{2}.$$

2. (a) The roots of the polynomial $x(x^2-2) = x(x-\sqrt{2})(x+\sqrt{2})$ are $x_1 = 0$, $x_2 = \sqrt{2}$ and $x_3 = -\sqrt{2}$. These are all simple and real, thus we aim determine constants $A, B, C \in \mathbb{R}$ such that

$$\begin{aligned}\frac{x-1}{x^2(x-2)} &= \frac{A}{x} + \frac{B}{(x-\sqrt{2})} + \frac{C}{(x+\sqrt{2})} \\ &= \frac{A\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right) + Bx\left(x+\sqrt{2}\right) + Cx\left(x-\sqrt{2}\right)}{x\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)} \\ &= \frac{(A+B+C)x^2 + \left(\sqrt{2}B - \sqrt{2}C\right)x - 2A}{x\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)}.\end{aligned}$$

By comparing coefficients, we construct the following system of equations

$$\begin{aligned}A + B + C &= 0 \\ \sqrt{2}B - \sqrt{2}C &= 1 \\ -2A &= -1,\end{aligned}$$

whose solution is given by

$$A = \frac{1}{2}, \quad B = \frac{\sqrt{2}-1}{4}, \quad C = -\frac{\sqrt{2}+1}{4}.$$

Finally, we compute

$$\begin{aligned}\int_2^3 \frac{x-1}{x\left(x^2-2\right)}\,\mathrm{d}x &= \int_2^3 \left(\frac{1}{2x} + \frac{\sqrt{2}-1}{4}\frac{1}{x-\sqrt{2}} - \frac{\sqrt{2}+1}{4}\frac{1}{x+\sqrt{2}}\right)\mathrm{d}x \\ &= \frac{1}{2}\ln(x) + \frac{\sqrt{2}-1}{4}\ln\left(x-\sqrt{2}\right) - \frac{\sqrt{2}+1}{4}\ln\left(x+\sqrt{2}\right)\Big|_{x=2}^{x=3} \\ &= \frac{1}{2}\ln\left(\frac{3}{2}\right) + \frac{\sqrt{2}-1}{4}\ln\left(\frac{3-\sqrt{2}}{2-\sqrt{2}}\right) - \frac{\sqrt{2}+1}{4}\ln\left(\frac{3+\sqrt{2}}{2+\sqrt{2}}\right).\end{aligned}$$

(b) The roots of the polynomial $(x+2)(x+3)^2$ are $x_1 = -2$ and $x_2 = -3$. Notice that x_1 is simple, whilst the second root has multiplicity 2. Therefore, we have to determine $A, B, C \in \mathbb{R}$ such that

$$\begin{aligned}\frac{x^2}{(x+2)(x+3)^2} &= \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{(x+3)^2} \\ &= \frac{A(x+3)^2 + B(x+2)(x+3) + C(x+2)}{(x+2)(x+3)^2}.\end{aligned}$$

In particular,

$$A(x+3)^2 + B(x+2)(x+3) + C(x+2) = x^2,$$

and so by choosing suitable values of x we determine the above constants:

by setting $x = -2$, we deduce that $A = 4$. This, together with the choice $x = -3$, tells us that $C = -9$. Finally, by inserting $x = 0$, we find $B = -3$.

The integral now evaluates to

$$\begin{aligned}\int_{-1}^{1} \frac{x^2}{(x+2)(x+3)^2}\,\mathrm{d}x &= \int_{-1}^{1} \left(\frac{4}{x+2} - \frac{3}{x+3} - \frac{9}{(x+3)^2}\right)\mathrm{d}x \\ &= 4\ln(x+2) - 3\ln(x+3) + \frac{9}{x+3}\bigg|_{x=-1}^{x=1} \\ &= 4\ln(3) - 3\ln(2) - \frac{9}{4},\end{aligned}$$

where we have used that $\ln 4 - \ln 2 = \ln(2 \cdot 2) - \ln 2 = \ln 2$.

3. (a) First, notice that $\cos x \sin x = \frac{1}{2}\sin 2x$. Then

$$\int \cos x \sin x\,\mathrm{d}x = \frac{1}{2}\int \sin 2x\,\mathrm{d}x = \frac{1}{4}\int \sin u\,\mathrm{d}u = -\frac{1}{4}\cos 2x + C_1,$$

where we use the substitution $u = 2x$.
Proceed using integration by parts with $f(x) = x$ and $g'(x) = \cos x \sin x$:

$$\begin{aligned}\int x\cos x \sin x\,\mathrm{d}x &= x \cdot \int \cos x \sin x\,\mathrm{d}x - \int \left(\int \cos x \sin x\,\mathrm{d}x\right)\mathrm{d}x \\ &= -\frac{x}{4}\cos 2x + \frac{1}{4}\int \cos 2x\,\mathrm{d}x\end{aligned}$$

Now, as above, use the substitution $u = 2x$ to compute the latter integral:

$$\frac{1}{4}\int \cos 2x\,\mathrm{d}x = \frac{1}{8}\int \cos u\,\mathrm{d}u = \frac{1}{8}\sin 2x + C_2.$$

Altogether, we have

$$\int x\cos x \sin x\,\mathrm{d}x = \frac{1}{8}\sin 2x - \frac{x}{4}\cos 2x + C_3.$$

(b) We directly compute

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2(x^2-1)} &= \int \frac{-(x^2-1) + \frac{1}{2}(x^3+x^2) - \frac{1}{2}(x^3-x^2)}{x^2(x+1)(x-1)}\,\mathrm{d}x \\ &= \int \left(\frac{-(x^2-1)}{x^2(x+1)(x-1)} + \frac{\frac{1}{2}(x^3+x^2)}{x^2(x+1)(x-1)} - \frac{\frac{1}{2}(x^3-x^2)}{x^2(x+1)(x-1)}\right)\mathrm{d}x \\ &= \int \left(-\frac{1}{x^2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right)\mathrm{d}x \\ &= \frac{1}{x} + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \\ &= \frac{1}{x} + \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C.\end{aligned}$$

(c) Set $u := \ln x$, such that $\frac{du}{dx} = \frac{1}{x}$. now

$$\int \frac{1}{x \ln x} = \int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C = \ln|\ln x| + C.$$

4. (a) Using the result of 1.(d), we compute

$$f_1(x) = \int_0^x \frac{e^{-\sqrt{t}}}{\sqrt{t}}\,\mathrm{d}t = -2e^{-\sqrt{t}}\Big|_{t=0}^{t=x} = 2 - 2e^{-\sqrt{x}}.$$

Thus,

$$f_1'(x) = (2 - 2e^{-\sqrt{x}})' = -2e^{-\sqrt{x}} \cdot (-\sqrt{x})' = \frac{e^{-\sqrt{x}}}{\sqrt{x}},$$

and we conclude that $f_1'(1) = e^{-1} = \frac{1}{e}$.

(b) We apply the result of 1.(d) again;

$$f_2(x) = \int_x^{x^2} \frac{e^{-\sqrt{t}}}{\sqrt{t}}\,\mathrm{d}t = -2e^{-\sqrt{t}}\Big|_{t=x}^{t=x^2} = 2e^{-\sqrt{x}} - 2e^{-x},$$

and therefore

$$f_2'(x) = (2e^{-\sqrt{x}} - 2e^{-x})' = 2e^{-x} - \frac{e^{-\sqrt{x}}}{\sqrt{x}}.$$

In particular, $f_2'(1) = 2e^{-1} - e^{-1} = e^{-1} = \frac{1}{e}$.

(c) Recall the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)\,\mathrm{d}t = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$$

given by the chain rule and the second fundamental theorem of calculus. In particular,

$$f_1'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{e^{-\sqrt{t}}}{\sqrt{t}}\,\mathrm{d}t = \frac{e^{-\sqrt{x}}}{\sqrt{x}},$$

and so $f_1'(1) = \frac{1}{e}$. Similarly

$$f_2'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} \frac{e^{-\sqrt{t}}}{\sqrt{t}}\,\mathrm{d}t = 2e^{-x} - \frac{e^{-\sqrt{x}}}{\sqrt{x}},$$

and $f_2'(1) = \frac{1}{e}$. Our results agree with the solutions to 4.(a) and 4.(b).

(d) We proceed as above and directly compute

$$g_1'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{dt}{t^3+1} = \frac{1}{x^3+1},$$

so $g_1'(2) = \frac{1}{9}$.

(e) Similarly to the computations above we find

$$g_2'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} \frac{dt}{t^3+1} = \frac{2x}{x^6+1},$$

and therefore $g_2'(2) = \frac{4}{65}$.

5. First calculate

$$\begin{aligned}\int f(t)\,\mathrm{d}t &= \int 25e^{-t}\,\mathrm{d}t + e^{-0.05t}\,\mathrm{d}t \\ &= -25e^{-t} - \frac{1}{0.05}e^{-0.05t} + C.\end{aligned}$$

We find the answer to the question by evaluating the corresponding definite integral from $t = 0$ to $t = 10$:

$$\int_0^{10} f(t)\,\mathrm{d}t = -25e^{-t} - \frac{1}{0.05}e^{-0.05t}\Big|_{t=0}^{x=10} \approx 33.$$

Therefore, in the first ten days, about 33 mm of water flows from the area affected by the weather.