Complex analysis

1. (a) $(3+5i)^2 = 9+30i-25 = -16+30i$.
   (b) $(-7+2i)(5-3i) = -35+21i+10i+6 = -29+31i$.
   (c) $|2+i| = \sqrt{4+1} = \sqrt{5}$, since $|z|^2 = z\bar{z} = x^2+y^2$ for $z = x+iy$.
   (d)
   $$\frac{i-1}{1+i} = \frac{(i-1)(1-i)}{2} = \frac{i-i^2-1+i}{2} = \frac{2i}{2} = i.$$
   (e)
   $$\frac{1-5i}{3i-1} = \frac{(1-5i)(-3i-1)}{10} = \frac{-3i-1-15+5i}{10} = -\frac{8}{5}+\frac{1}{5}i.$$
   (f)
   $$\frac{1-i}{1+i}+2-i = -\frac{i-1}{1+i}+2-i = -i+2-i = 2-2i,$$
   where we apply the solution of 1.(a) for second equality.

2. Every complex number z has a unique representation
   $$z = re^{i\varphi} = r(\cos\varphi + i\sin\varphi),$$
   for $r \in \mathbb{R}_+$ and $-\pi < \varphi \leq \pi$.
   (a) We compute $r = |z| = \sqrt{5^2+1} = \sqrt{26}$, and thus by comparing real and imaginary parts,
   $$\cos\varphi = -\frac{5}{\sqrt{26}} < 0, \qquad \sin\varphi = \frac{1}{\sqrt{26}} > 0.$$
   The angle φ is then given by $\varphi = \pi + \arctan\frac{y}{x} = \pi + \arctan\frac{-1}{5}$.
   (b) Similarly to the calculations above we find $r = |z| = \sqrt{4+9} = \sqrt{13}$, so
   $$\cos\varphi = \frac{2}{\sqrt{13}} > 0, \qquad \sin\varphi = -\frac{3}{\sqrt{13}} < 0.$$
   In particular, $\varphi = \arctan\frac{y}{x} = \arctan\frac{-3}{2}$.

3. (a) We may rewrite $z^2+4z+12-6i = 0$ as $(z+2)^2 = -8+6i$; after setting $z+2 = x+iy$ we compare real and imaginary parts to find
   $$x^2 - y^2 = -8$$
   $$2xy = 6.$$
   The second equation gives $y = \frac{3}{x}$ and thus the first one becomes
   $$t^2+8t-9 = 0,$$
   where we have substituted $x^2 = t$. Solving this equation we find $t = -9$ or $t = 1$. Since only real solutions are valid, we conclude that $x = \pm 1$ and hence $y = \pm 3$. The final solutions of the original equation are therefore $z = -1+3i$ and $z = -3-3i$.

(b) In order to solve this equation, we first determine the polar representation $re^{i\phi}$ of $-4\sqrt{3}-4i$, with $-\pi < \varphi \leq \pi$. Namely,

$$r = |-4\sqrt{3}-4i| = \sqrt{16\cdot 3+16} = \sqrt{64} = 8,$$

and using the same procedure as in q.2, we compute φ to be

$$\varphi = -\pi + \arctan\frac{-4}{-4\sqrt{3}} = -\pi + \arctan\frac{\sqrt{3}}{3} = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}.$$

If now $z = se^{i\vartheta}$ is a solution of the equation

$$z^6 = 8e^{-i\frac{5\pi}{6}},$$

it follows that

$$s = 8^{1/6} = \sqrt{2} \quad \text{and} \quad \vartheta = \frac{-5\pi}{36} + k\cdot\frac{2\pi}{6}, \quad k\in\mathbb{Z}.$$

We conclude that there are in total six different solutions to the initial equation $z^6 = -4\sqrt{3}-4i$:

$$z_0 \overset{k=0}{=} \sqrt{2}e^{-5i\pi/36} \qquad z_1 \overset{k=1}{=} \sqrt{2}e^{7i\pi/36} \qquad z_2 \overset{k=2}{=} \sqrt{2}e^{19i\pi/36}$$
$$z_3 \overset{k=3}{=} \sqrt{2}e^{31i\pi/36} \qquad z_4 \overset{k=4}{=} \sqrt{2}e^{43i\pi/36} \qquad z_5 \overset{k=5}{=} \sqrt{2}e^{55i\pi/36};$$

for $k=6$ we compute

$$z_6 \overset{k=6}{=} \sqrt{2}e^{67i\pi/36} = \sqrt{2}e^{67i\pi/36-2i\pi} = z_0.$$

(c) As above, we proceed by first finding the polar representation $re^{i\varphi}$ of the equation $z^5 = (1+i)(-1+\sqrt{3}\,i)(\sqrt{3}-i)$. Notice that

$$\begin{aligned} z^5 &= (1+i)(-1+\sqrt{3}\,i)(\sqrt{3}-i) \\ &= (1+i)\cdot i\cdot(\sqrt{3}+i)(\sqrt{3}-i) \\ &= 4i\cdot(1+i) = 4i-4. \end{aligned}$$

In particular,

$$r = \sqrt{16+16} = \sqrt{32} = (\sqrt{2})^5,$$

and the argument φ is

$$\varphi = \pi + \arctan(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

If now $z = se^{i\vartheta}$ is a solution of the equation

$$z^5 = (\sqrt{2})^5 e^{3i\pi/4},$$

we must have

$$s = \sqrt{2} \quad \text{and} \quad \vartheta = \frac{3\pi}{20} + k \cdot \frac{2\pi}{5}, \quad k \in \mathbb{Z}.$$

We conclude that there are in total five different solutions to the initial equation $z^5 = (1+i)(-1+\sqrt{3}\,i)(\sqrt{3}-i)$:

$$z_0 \overset{k=0}{=} \sqrt{2}e^{3i\pi/20} \qquad z_1 \overset{k=1}{=} \sqrt{2}e^{11i\pi/20} \qquad z_2 \overset{k=2}{=} \sqrt{2}e^{19i\pi/20}$$
$$z_3 \overset{k=3}{=} \sqrt{2}e^{27i\pi/36} \qquad z_4 \overset{k=4}{=} \sqrt{2}e^{35i\pi/36}.$$

4. (a) By writing $z = x + iy$ we find $\text{Re} = \{(x, y) \mid x > 2\}$, which corresponds to a half plane.

   (b) The condition $|z| < 3$ is equivalent to $x^2 + y^2 < 9$ for $z = x + iy$, and hence our set is an open disc of radius 3 centred at the origin.

   (c) Write again $z = x + iy$; then the condition $|z-1| < |z+1|$ is equivalent to $(x-1)^2 + y^2 < (x+1)^2 + y^2$. After cancellations, we simply end up with the inequality $x > 0$, corresponding to an open half plane.

5. (a) We directly compute

$$\begin{aligned} z \cdot w &= rs(\cos\vartheta\cos\varphi - \sin\vartheta\sin\varphi + i(\sin\vartheta\cos\varphi + \cos\vartheta\sin\varphi)) \\ &= rs(\cos(\vartheta+\varphi) + i\sin(\vartheta+\varphi)), \end{aligned}$$

   and therefore $\arg(z \cdot w) = \arg z + \arg w$.

   (b) De Moivre's theorem clearly holds for $n = 1$, and for $n = 2$ the above computation immediately gives us $\arg(z^2) = 2\arg z$ for any complex number z. Suppose now that $\arg(z^n) = n \arg z$, for $n \geq 2$. Then

$$\arg(z^{n+1}) = \arg(z \cdot z^n) = \arg(z) + n\arg(z) = (n+1)\arg z,$$

   where we apply the induction hypothesis for the second equality. This in particular shows that $\arg(z^n) = n \arg z$, for any $n \geq 1$.

   Set $z = \cos\varphi + i\sin\varphi$ and notice that $\arg z^n = n\varphi$, since $\arg z = \varphi$. To complete the proof of De Moivre's Theorem, observe that $|z| = \cos^2\varphi + \sin^2\varphi = 1$, thus also $|z^n| = 1$. In particular, $z^n = \cos n\varphi + i\sin n\varphi$.

   (c) Apply De Moivre's Theorem to the case $n = 3$:

$$\begin{aligned} (\cos\varphi + i\sin\varphi)^3 &= \cos^3\varphi + 3i\cos^2\varphi\sin\varphi + 3i^2\cos\varphi\sin\varphi + i^3\sin^3\varphi \\ &= \cos 3\varphi + i\sin 3\varphi, \end{aligned}$$

   By comparing real and imaginary parts, and using that $i^2 = -1$, $i^3 = -i$, we conclude that

$$\begin{aligned} \cos 3\varphi &= \cos^3\varphi - 3\cos\varphi\sin^2\varphi = 4\cos^3\varphi - 3\cos\varphi, \\ \sin 3\varphi &= 3\cos^2\varphi\sin\varphi - \sin^3\varphi = 3\sin\varphi - 4\sin^3\varphi. \end{aligned}$$

6. (a) Every $z \in \mathbb{C}$ has n distinct roots of order n, which correspond (in the complex plane) to the vertices of a regular n-agon inscribed in the circle of radius $|z|^{1/n}$ centered at the origin. When $z = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$, then the roots of order n of z are

$$\rho^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right) \text{ for } k = 0, 1, \dots, n-1.$$

The square roots of $z = -1 - i = \sqrt{2}(\cos 5\pi/4 + i\sin 5\pi/4)$ are

$$z_1 = (2^{\frac{1}{2}})^{\frac{1}{2}}\left(\cos\left(\frac{\frac{5\pi}{4}}{2}\right) + i\sin\left(\frac{\frac{5\pi}{4}}{2}\right)\right) = 2^{\frac{1}{4}}\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right),$$
$$z_2 = (2^{\frac{1}{2}})^{\frac{1}{2}}\left(\cos\left(\frac{\frac{5\pi}{4}+2\pi}{2}\right) + i\sin\left(\frac{\frac{5\pi}{4}+2\pi}{2}\right)\right) = 2^{\frac{1}{4}}\left(\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}\right),$$

We could also have argued as follows: the equation $(x+iy)^2 = -1 - i$ is equivalent to the system

$$\begin{cases} x^2 - y^2 = -1, \\ 2xy = -1, \end{cases}$$

which admits solutions

$$z = \pm\left(\left(\frac{\sqrt{2}-1}{2}\right)^{\frac{1}{2}} - \frac{i}{2}\left(\frac{2}{\sqrt{2}-1}\right)^{\frac{1}{2}}\right)$$

which coincide with z_1 and z_2.

(b) The trigonometric form of $z = -8$, is $z = 8(\cos\pi + i\sin\pi)$. Then

$$z_1 = 8^{\frac{1}{3}}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3},$$
$$z_2 = 8^{\frac{1}{3}}(\cos\pi + i\sin\pi) = -2,$$
$$z_3 = 8^{\frac{1}{3}}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 1 - i\sqrt{3},$$

7. If $z = a + ib$, $a, b \in \mathbb{R}$ then $z^2 \in \mathbb{R}$ if and only if $a^2 - b^2 + 2iab$ is a real number. That is, if and only if $ab = 0$. Hence $z^2 \in \mathbb{R}$ if and only if $z \in \mathbb{R}$ $(b = 0)$ or if z is a pure imaginary number $(a = 0)$.