DIFFERENTIAL MULTIVARIABLE CALCULUS

1. (a) We differentiate with respect to $x$ first, such that

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}(x\sin(y)+e^y) = \frac{\partial}{\partial y}\sin y = \cos y.$$

(b) We should differentiate with respect to $y$ first:

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}(\tfrac{1}{x}) = 0.$$

(c) Differentiating in $x$ first gives

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}(y+\tfrac{x}{y}) = \frac{\mathrm{d}}{\mathrm{d}y}\frac{1}{y} = -\frac{1}{y^2}.$$

(d) With respect to $x$ first we have

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}(y+x^2y+4y^3-\ln(y^2+1)) = \frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x.$$

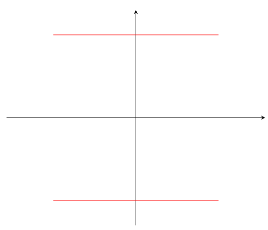
(e) Differenatiate with respect to $y$ first such that

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}x^2+5xy+\sin(x)+7e^x = \frac{\mathrm{d}}{\mathrm{d}x}5y = 5.$$

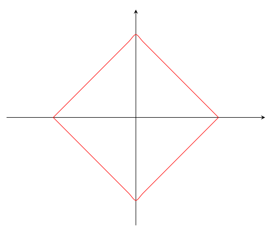
(f) With respect to $y$ first gives

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}(x\ln(xy)) = \frac{\partial}{\partial x}(x\cdot\tfrac{x}{xy}) = \frac{1}{y}.$$

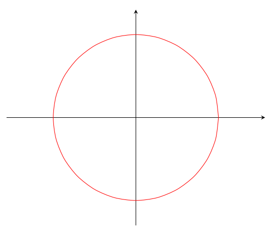
2. (a) Two parallel lines $y=\pm\sqrt{c}$:



(b) Lozenge centred at the origin:



(c) Circle centred at the origin:



3. The gradient of a function $f(x, y)$ is given by

$$\nabla f(x, y) = (f_x, f_y),$$

and so we simply need to calculate $f_x = \frac{\partial}{\partial x} f(x, y)$ and $f_y = \frac{\partial}{\partial y} f(x, y)$.

   (a) $f_x = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 - 9} = \frac{x}{\sqrt{x^2+y^2-9}}$ and $f_y = \frac{\partial}{\partial y}\sqrt{x^2 + y^2 - 9} = \frac{y}{\sqrt{x^2+y^2-9}}$.

   (b) $f_x = \frac{\partial}{\partial x} xy = y$ and $f_y = \frac{\partial}{\partial y} xy = x$.

   (c) $f_x = \frac{\partial}{\partial x}(x^3 + 3(x^2 - y^2) - 3) = 3x^2 + 6x$ and $f_y = \frac{\partial}{\partial y}(x^3 + 3(x^2 - y^2) - 3) = -6y$.

4. The intersection of the surface with the plane $x = 2$ is the curve $z = \arctan(2y)$. To find the tangent line, we first work in the plane $x = 2$, before extending to three dimensional space.

   *In the plane:* the slope of the tangent line is given by $m = \frac{\partial z}{\partial y} = \frac{2}{1+4y^2}$, which evaluates to $m = \frac{2}{1+1} = 1$ at $y = \frac{1}{2}$. Recall that the equation of a straight line with slope $m$ passing through a point $(y_0, z_0)$ is given by $z - z_0 = m(y - y_0)$. Since our tangent line must pass through the point $(y, z) = (\frac{1}{2}, \frac{\pi}{4})$, we arrived at $z = y - \frac{1}{2} + \frac{\pi}{4}$.

   *In three dimensions:* to find a correct description in three dimensions, use that the first coordinate $x$ is constantly equal to 2. In particular, our tangent line is given by $\left\{\left(2, y, y - \frac{1}{2} + \frac{\pi}{4}\right) : y \in \mathbb{R}\right\}$. Be aware that there are many other possible descriptions (parametrisations); for example $\left(2, z + \frac{1}{2} - \frac{\pi}{4}, z\right)$ for $z \in \mathbb{R}$ or $\left(2, \frac{1}{2} + t, \frac{\pi}{4} + t\right)$ for $t \in \mathbb{R}$.

5. The volume of the box is given by $V(a, b, c) = abc$; the rate of change at any specific time $t$ is therefore

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial(abc)}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\partial(abc)}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}t} + \frac{\partial(abc)}{\partial c}\frac{\mathrm{d}c}{\mathrm{d}t} = bc\frac{\mathrm{d}a}{\mathrm{d}t} + ca\frac{\mathrm{d}b}{\mathrm{d}t} + ab\frac{\mathrm{d}c}{\mathrm{d}t}.$$

   Evaluating this at time $t_0$ gives $V'(a, b, c)|_{t=t_0} = 6 \cdot 1 + 3 \cdot 1 + 2 \cdot (-3) = 3$.

   The surface area is $A(a, b, c) = 2(ab + bc + ca)$; thus the rate of change is given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2(b + c)\frac{\mathrm{d}a}{\mathrm{d}t} + 2(c + a)\frac{\mathrm{d}b}{\mathrm{d}t} + 2(a + b)\frac{\mathrm{d}c}{\mathrm{d}t}.$$

   Evaluating at $t_0$, we get $A'(a, b, c)|_{t=t_0} = 0$.

   Finally, the length of an internal diagonal is given by $D(a, b, c) = \sqrt{a^2 + b^2 + c^2}$. By using the chain rule, we compute

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{2\sqrt{a^2 + b^2 + c^2}}\left(2a\frac{\mathrm{d}a}{\mathrm{d}t} + 2b\frac{\mathrm{d}b}{\mathrm{d}t} + 2c\frac{\mathrm{d}c}{\mathrm{d}t}\right);$$

since the term in the brackets is $2(1+2-9)<0$ at time $t_0$, the diagonals must be decreasing in length.

6. Since $y = 12 - x$, we can reduce this question to a problem in one variable. Set $f(x) = x^2 y = x^2(12-x)$. The minimum of $f(x)$ with $x$ restricted to $[0, 12]$ can occur at the endpoints of this interval, or when $f'(x) = 0$. Since $f'(x) = 24x - 3x^2$, the latter case corresponds to $x = 8$.

   We compute $f(0) = f(12) = 0$ and $f(8) = 256$; thus the minimum $f(x) = 0$ is attained for $x = 0$, $y = 12$ and for $x = 12$, $y = 0$.

7. We work with the function

$$F(x, y, z) = x^2 + y^2 + z^2$$

   and note that we don't have to have a zero on one side of the equal sign. All that we need is a constant. To finish this problem out we simply need the gradient evaluated at the point.

$$\begin{aligned} &\nabla F = (2x, 2y, 2z), \\ &\nabla F(1, -2, 5) = (2, -4, 10) \end{aligned}$$

   The tangent plane is then,

$$2(x-1) - 4(y+2) + 10(z-5) = 0$$

   and the normal line is

$$r(t) = (1, -2, 5) + t(2, -4, 10) = (1 + 2t, -2 - 4t, 5 + 10t)$$