Directional Derivatives and Differential Equations of First Order

1. (a) To use the chain rule, we need the four quantities $dz/dx, dz/dy, dx/dt$, and dy/dt:
$$\frac{dz}{dx} = 8x,\ \frac{dx}{dt} = \cos(t),\ \frac{dz}{dy} = 6y,\ \text{and}\ \frac{dy}{dt} = -\sin(t)$$
Now, we substitute each of these into Equation $dz/dt = dz/dx \cdot dx/dt + dz/dy \cdot dy/dt$ to obtain
$$\frac{dz}{dt} = 8x\cos(t) + 6y - \sin(t)$$
This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin t$ and $y(t) = \cos t$. We obtain
$$\frac{dz}{dt} = 2\sin(t)\cos(t)$$
(b) To use the chain rule, we need again the four quantities $dz/dx, dz/dy, dx/dt$, and dy/dt:
$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2-y^2}},\ \frac{dx}{dt} = 2e^{2t},\ \frac{dz}{dy} = \frac{-y}{\sqrt{x^2-y^2}},\ \text{and}\ \frac{dy}{dt} = -e^{-t}$$
Again we substitute in $dz/dt = dz/dx \cdot dx/dt + dz/dy \cdot dy/dt$ to obtain
$$\frac{dz}{dt} = \frac{2xe^{2t} - ye^{t}}{\sqrt{x^2-y^2}}$$
To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$. Therefore,
$$\frac{dz}{dt} = \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}$$

2. To use the chain rule for two variables, we need the six quantities dz/dx, dz/dy, dx/du, dx/dv, dy/du, and dy/dv:
$$\frac{dz}{dx} = 6x - 2y,\ \frac{dz}{dy} = -2x + 2y,\ \frac{dx}{du} = 3,$$
$$\frac{dx}{dv} = 2,\ \frac{dy}{du} = 4,\ \frac{dy}{dv} = -1$$
Then we using the formula and substituting with $x(u,v) = 3u+2v$ and $y(u,v) = 4u - v$, we obtain
$$\begin{aligned}\frac{dz}{du} &= \frac{dz}{dx}\cdot\frac{dx}{du} + \frac{dz}{dy}\cdot\frac{dy}{du}\\ &= 10x + 2y\\ &= 38u + 18v.\end{aligned}$$

Similarly for dz/dv:

$$\frac{dz}{dv} = 18u + 34v.$$

3. We first need the gradient

$$\nabla f = (-y\sin(xy), -x\sin(xy)).$$

Also recall that we need to make sure that the direction vector is a unit vector. So let's convert it to a unit vector.

$$|\overrightarrow{v}| = \sqrt{3^2 + (-4)^2} = 5, \text{ so that } \overrightarrow{u} = \frac{\overrightarrow{v}}{|\overrightarrow{v}|} = \left(\frac{3}{5}, \frac{-4}{5}\right).$$

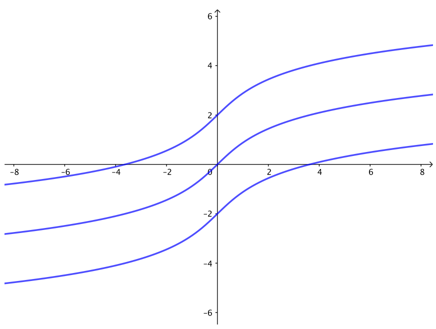
The directional derivative is then

$$\begin{aligned} D_{\overrightarrow{u}} f &= (-y\sin(xy), -x\sin(xy)) \cdot \left(\frac{3}{5}, \frac{-4}{5}\right) \\ &= \frac{-3y}{5}\sin(xy) + \frac{-4x}{5}\sin(xy) \end{aligned}$$

4. (a) Simply separate and integrate:

$$\int \mathrm{d}y = y = \int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \mathrm{arcsinh}x + C,$$

for some constant $C \in \mathbb{R}$. This is the general solution; below is a plot for various values of C.



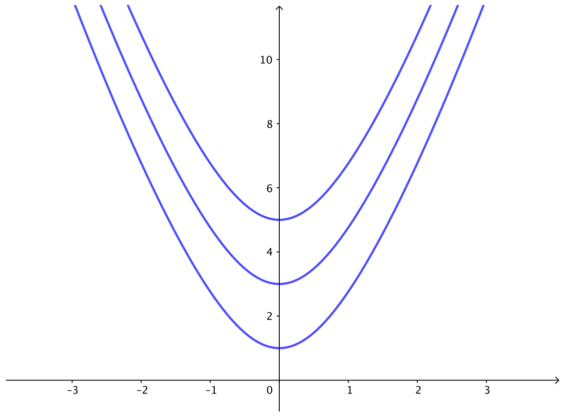
The solution curves all have the same shape. Variations in C shift them along the y-axis.

(b) Separate and integrate, using the substitution $z = 1 + x^2$:

$$\int \mathrm{d}y = y = \int \frac{4x}{(1+x^2)^{1/3}}\,\mathrm{d}x = \int \frac{2}{z^{1/3}}\,\mathrm{d}z = 3z^{2/3} + C,$$

and so the general solution is given by $y = 3(1+x^2)^{2/3} + C$. A sketch of the solution for varying values of C is shown below.

Again, the constant C simply shifts the position of the solution curves.



5. (a) Solving the given differential equation by separation of variables leads to $\ln y = \ln x + C_1$, and so $y = Cx$ for some constant $C = e^{C_1}$. These are all lines through the origin.

   (b) The curves orthogonal to lines through the origin are circles centred at the origin.

   (c) The orthogonal curves have equations $x^2 + y^2 = r^2$, where r is the radius. Differentiating this equation with respect to x, results in

$$2x + 2y\frac{dy}{dx} = 0,$$

   or $\frac{dy}{dx} = -\frac{x}{y}$, as required.

   In particular, the orthogonal circles have slope $-\frac{x}{y}$ at each point (x, y), which is precisely the negative reciprocal of $\frac{y}{x}$, the slope of the integral lines.

6. (a) The first order differential equation we need to solve is given by

$$y' = 0,1 \cdot y \cdot (4 - 2x).$$

   First separate the variables such that

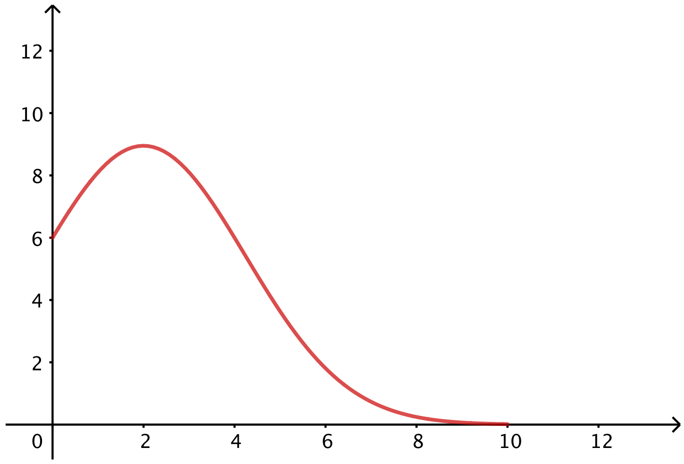
$$\frac{\mathrm{d}y}{y} = 0,1 \cdot (4 - 2x)\,\mathrm{d}x,$$

   and integrate to arrive at

$$\ln|y| = 0,1 \cdot (4x - x^2) + C_1,$$

   for some constant $C_1 \in \mathbb{R}$. Solving for y gives

$$y = Ce^{0,1\cdot(4x-x^2)},$$



where $C = e_1^C$. Since $y(0) = 6$, we find that $C = 6$; in other words,

$$y = 6e^{0,1\cdot(4x-x^2)}.$$

(b) The graph of $y = 6e^{0,1\cdot(4x-x^2)}$ is displayed below. The number of bacteria is first increasing; however, after the effects of the penicillin treatment become more significant, the illness disappears.

(c) Setting the differential equation equal to zero gives

$$0,1 \cdot y \cdot (4 - 2x) = 0,$$

and so $y' = 0$ occurs when $x = 2$. The highest number of bacteria in the patient's system is therefore $8,95$ millions, after two days.